BEHAVIOR OF PERIODIC SOLUTIONS OF VISCOUS CONSERVATION LAWS UNDER LOCALIZED AND NONLOCALIZED PERTURBATIONS

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Abstract. We establish nonlinear stability and asymptotic behavior of traveling periodic waves of viscous conservation laws under localized perturbations or nonlocalized perturbations asymptotic to constant shifts in phase, showing that long-time behavior is governed by an associated second-order formal Whitham modulation system. A key point is to identify the way in which initial perturbations translate to initial data for this formal system, a task accomplished by detailed estimates on the linearized solution operator about the background wave. Notably, our approach gives both a common theoretical treatment and a complete classification in terms of "phase-coupling" or "-decoupling" of general systems of conservation or balance laws, encompassing cases that had previously been studied separately or not at all. At the same time, our refined description of solutions gives the new result of nonlinear asymptotic stability with respect to localized perturbations in the phase-decoupled case, further distinguishing behavior in the different cases. An interesting technical aspect of our analysis is that for systems of conservation laws the Whitham modulation description is of system rather than scalar form, as a consequence of which renormalization methods such as have been used to treat the reaction-diffusion case in general do not seem to apply.

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Contents

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1. Introduction

One of the triumphs in recent years in the dynamical study of partial differential equations (PDE) has been the development of a rigorous theory of modulation of periodic traveling waves in optics, pattern-formation, and other equations, both illuminating and expanding on formal predictions made by WKB-type expansion much earlier on, as for example in [68, 18]. Among many other results, we mention in particular the resolution in [59, 60, 61] using Bloch transform/renormalization techniques of the then 30-year open problem of stability of periodic reaction-diffusion waves with respect to localized perturbations[1] and, under nonlocalized perturbations, the rigorous verification in [15] using related techniques of the associated second order ("diffusive") formal WKB expansion in various settings, in particular in the small-wavelength limit. Most recently, the WKB expansion has been verified for solutions of reaction-diffusion equations in the long-time limit, in [57] by methods related to those of [59, 60, 61, 15] and in [25, 26] by rather different techniques originating from the study of conservation laws [29, 30, 74, 75].

From these analyses emerges the clear picture of asymptotic behavior as dominated by a single critical mode of the linearized equations, corresponding to translational invariance of the underlying equations, that is governed approximately by the phase equation of the formal WKB approximation: (the integral of) a scalar convected Burgers equation. However, there are many physically interesting applications to which this well-developed theory does not apply. Specifically, when there exist conserved quantities, whether deriving from Hamiltonian structure/symmetries of the equations,[2] or, as in the case of parabolic conservation laws considered here, simply from divergence form of the equations/conservation of mass, then there exist additional critical modes, and the formal

[1]Verifying formal predictions and rigorous spectral descriptions of [17].
[2]As for example for the Korteweg–de Vries (KdV) equation [68, 62, 31, 32] or Euler-Korteweg system [9].

WKB prediction becomes that of a more complicated *hyperbolic–parabolic system of conservation laws* rather than the scalar convected Burgers equation of the reaction-diffusion case.

Perhaps the best-known example of such a model is the Kuramoto–Sivashinsky equation, for which the formal asymptotic description of behavior via a hyperbolic–parabolic system of conservation laws was pointed out already in [18] under the alternative form of a damped scalar wave equation (the "viscoelastic behavior" of the title). Further examples arise in the modeling of viscoelasticity with strain-gradient effects, inclined thin-film flow, and Bénard–Marangoni or surfactant-driven Marangoni flow; see Section 1.4 and Appendix D.

Despite the physical motivation coming from such examples, until very recently there was no rigorous analysis of nonlinear stability or behavior in this (system) case. Indeed, as discussed in Remark 2.2, the renormalization techniques of the asymptotically scalar reaction-diffusion case in the presence of multiple characteristic speeds (linear group velocities) appear to break down. Using a technically rather different set of techniques, nonlinear stability under localized perturbations has now been shown for such systems in [53][3] [29, 30, 33, 3, 4] in great generality, in particular resolving the longstanding open problem of nonlinear stability of spectrally stable Kuramoto–Sivashinsky waves, dating back to the numerical confirmation in [18] of existence of bands in parameter space of spectrally stable waves. However, up to now, asymptotic behavior has not been determined in this more complicated, system, case even for localized perturbations.

More, as discussed in [29, 30], there was some question in this case precisely what behavior one might expect. Specifically, one-dimensional nonlinear modulational stability under localized perturbations of spectrally stable periodic traveling wave solutions of viscous conservation laws was shown in [29] and [30] in two different cases, depending roughly on whether or not the wave speed is stationary to first order along the manifold of nearby periodic solutions. These two analyses were motivated by a common connection observed by Serre [62] to an associated formal Whitham averaged system obtained by WKB approximation. However, despite this shared heuristic description, the authors observed some puzzling asymmetries in the results obtained; see, for example, the discussion in [30, Section 1.3] on the varying linearized and nonlinear decay rates obtained in these different cases under localized perturbations. In particular, the nonlinear decay rate obtained for localized perturbations in the stationary case was slower than what might be guessed from the formal Whitham approximation with zero initial phase modulation; however, it was left as an open problem whether this intuitive initialization was correct, or whether localized initial perturbations could excite the phase mode through nonlinear interaction in some way.

Here, we sharpen and extend these previous results ([29, 30]) in several ways, in particular allowing more general, nonlocalized, perturbations and rigorously identifying time-asymptotic behavior as agreeing to leading order with the solution of the formal Whitham system with appropriately prescribed initial data. Our analysis loosely follows, and also greatly extends, the approach of [25, 26] in the reaction-diffusion case; as noted earlier, we do not see a way to apply here the more familiar techniques of [59, 60, 61, 15, 57]. In the process, we explain the asymmetries observed in [29, 30] as connected with the different ways that initial data align with characteristic modes for the common Whitham system governing large-time asymptotics.

A striking consequence of our results is that spectrally stable waves about which wave speed depends to first order on wave number alone are not only boundedly nonlinearly stable, but *asymptotically stable* with respect to localized perturbations. This resolves a question brought up early on in [50, 51, 62] that was left open in the analysis of [29, 30]. On the other hand, with respect to nonlocalized perturbations, waves in the two different cases behave essentially alike.

To put things another way, we show that the case that the part of the Whitham system corresponding to phase perturbations decouples from the rest of the Whitham equations yields decay

[3] Concerning the more tractable (since faster-decaying) three and higher dimensional case.

rates exactly corresponding to those of the (scalar[4] Whitham equation) reaction-diffusion case, both for localized and nonlocalized perturbations. Indeed, we find that reaction-diffusion and conservation laws can be put in a common framework

$$u_t + f(u)_x + g(u) = (B(u)u_x)_x, \quad u \in \mathbb{R}^n,$$

consisting of a continuum of models, with $f \equiv 0$ corresponding to the reaction-diffusion case and $g \equiv 0$ to the conservative case, for which a complete classification of behavior can be obtained.

This analysis puts the conservative theory now on a par with that of the reaction-diffusion case, at least as far as time-asymptotic stability and asymptotic behavior. It is an interesting open problem to reproduce in the conservative case a small-wavelength description as obtained in [15] for the reaction-diffusions case. See [48, 49] for some preliminary results in this direction.

1.1. **Slow modulation behavior.** We begin by emphasizing some insights gained from the WKB approximation process, which requires, first, a description of nearby periodic traveling waves. For definiteness/clarity of exposition, we restrict to the simplest case of a semilinear second-order parabolic system of conservation laws. However, our analysis extends with little change to the quasilinear $2r$-parabolic or (under appropriate structural conditions as in [35, 72, 74]) the symmetrizable hyperbolic-parabolic case; see [3, 4, 5, 33] for related analyses in these and more general situations. Examples include periodic solutions of the equations of one-dimensional viscoelasticity with strain-gradient effects [50, 7, 69] and of the Kuramoto–Sivashinsky equations [34, 63, 18] and Saint-Venant equations [16, 1, 46, 47] modeling inclined thin film flow. We discuss in Appendix D the changes needed to handle these interesting physical applications.

Consider a periodic traveling-wave solution of a parabolic or "viscous" system of conservation laws

$$u_t + f(u)_x = u_{xx},$$

u, f valued in $\mathbb{R}^n$, $x, t \in \mathbb{R}$, or, equivalently, a standing-wave solution $u(x,t) = \bar{U}(x)$ of

$$u_t + \bar{k}(f(u)_x - \bar{c}u_x) = \bar{k}^2 u_{xx}, \tag{1.1}$$

where $\bar{c}$ is the speed of the original traveling wave, and the wave number $\bar{k}$ is chosen so that

$$\bar{U}(x+1) = \bar{U}(x). \tag{1.2}$$

Integrating the traveling-wave equation $\bar{k}\bar{U}'' = f(\bar{U})' - \bar{c}\,\bar{U}'$ obtained by substituting $u(x,t) = \bar{U}(x)$ in (1.1), we obtain

$$\bar{k}\bar{U}' = f(\bar{U}) - \bar{c}\,\bar{U} + \bar{q}, \tag{1.3}$$

where $\bar{q} \in \mathbb{R}^n$ is a constant of motion. Setting $\bar{U}_0 := \bar{U}(0)$, we have evidently $(2n+2)$ parameters $(\bar{k}, \bar{c}, \bar{U}_0, \bar{q})$ determining candidates for periodic solutions, and n constraints $\bar{U}(1) = \bar{U}_0$, suggesting, in the absence of additional special structure[5] that the set of nearby periodic solutions form a manifold of dimension $n+2$. Denoting by $\bar{M} := \int_0^1 \bar{U}(x)dx$ the mean of $\bar{U}$, we make the genericity assumptions:

- **(H1)** $f \in C^K(\mathbb{R}^n)$ for some $K \geq 4$.
- **(H2)** Up to translation, the set of 1-periodic solutions of (1.1) (with M, k replacing $\bar{M}, \bar{k}$) in the vicinity of $\bar{U}$, $M = \bar{M}$, $k = \bar{k}$, forms a smooth $(n+1)$-dimensional manifold

$$\{\ (U(M,k;\cdot), c(M,k)) \mid (M,k) \in \Omega\ \} = \left\{\ (U^{M,k}(\cdot), c(M,k)) \ \middle|\ (M,k) \in \Omega\ \right\} \tag{1.4}$$

[4]See Remark 1.2 below.

[5]For example, Hamiltonian structure or existence of additional conserved quantities other than $\bar{q}$ [62, 30, 9].

where Ω is some open subspace of $\mathbb{R}^{n+1}$ containing $(\bar{M},\bar{k})$ and the role of M is defined implicitly by

$$M := \int_0^1 U^{M,k}(x)dx. \tag{1.5}$$

Then, the formal approximate solution of $u_t+\bar{k}(f(u)_x-\bar{c}u_x)-\bar{k}^2u_{xx}=0$ obtained by a nonlinear WKB expansion, as derived to varying orders of accuracy in [62, 52, 48, 49], is

$$u(x,t)\approx U^{(\mathcal{M},\kappa)(x,t)}(\Psi(x,t)), \tag{1.6}$$

where the mean $\mathcal{M}$ and wave number $\kappa := \bar{k}\Psi_x$ satisfy the Whitham equations

$$\begin{aligned} \mathcal{M}_t+\bar{k}(F-\bar{c}\mathcal{M})_x &= \bar{k}^2(d_{11}\mathcal{M}_x+d_{12}\kappa_x)_x,\\ \kappa_t+\bar{k}(-\omega-\bar{c}\kappa)_x &= \bar{k}^2(d_{21}\mathcal{M}_x+d_{22}\kappa_x)_x, \end{aligned} \tag{1.7}$$

an enlarged, $(n+1)\times(n+1)$ system of viscous conservation laws, where $\omega(\mathcal{M},\kappa)=-\kappa c(\mathcal{M},\kappa)$ denotes time frequency, $F(\mathcal{M},\kappa):=\int_0^1 f(U^{\mathcal{M},\kappa}(x))dx$ mean flux, and $d_{ij}(\mathcal{M},\kappa)$ are determined by higher-order corrections as described in [48, 49]. For convenience of the reader, we recall these derivations in Appendix B. The phase Ψ may be recovered through the solution of

$$\Psi_t=\omega(\mathcal{M},\kappa)+\bar{c}\,\kappa+\bar{k}d_{21}(\mathcal{M},\kappa)\mathcal{M}_x+\bar{k}d_{22}(\mathcal{M},\kappa)\kappa_x. \tag{1.8}$$

Equations (1.7) may be recognized[6] as a hyperbolic-parabolic system of conservation laws of general form $w_t+g(w)_x=(B(w)w_x)_x$, of which the scalar Burgers equation seen for reaction diffusion equations is a special case. For localized perturbations of a constant state $(\bar{M},\bar{k})$, that is $(\mathcal{M},\kappa)=(\bar{M},\bar{k})+(M,k)$ with (M,k) localized, of the type we consider here (corresponding to the kind of perturbation we allow for the original wave $U^{\bar{M},\bar{k}}$) the dynamics of such systems consists to lowest order of a superposition of Burgers "diffusion waves," or distorted Gaussians, directed along the characteristic directions associated with the first-order part $w_t+dg(w)w_x=0$ and traveling with associated characteristic speeds, plus their nonlinear interactions. In the generic case that characteristic speeds are distinct, these nonlinear coupling terms are smaller than the component diffusion waves; see Appendix B.2 for further description. In the case that characteristic speeds are complex, that is, hyperbolicity fails, constant solutions are time-exponentially unstable, and the above description breaks down; however, as noted below, this corresponds to the situation of an unstable background periodic wave, so is still in some sense consistent with behavior.

Thus, we expect the picture of (M,k) as approximately a superposition of signals with a roughly Gaussian distribution propagating at different speeds in the different characteristic directions of the first-order part of the Whitham equation. The phase $\Psi=\mathrm{Id}_{\mathbb{R}}+\psi$, as anti-derivative of $\kappa/\bar{k}=1+k/\bar{k}$, thus appears roughly as $\mathrm{Id}_{\mathbb{R}}$ plus a superposition ψ of distorted error-functions traveling with different speeds. Together, this gives a description *via* (1.6) of the expected behavior about a perturbed periodic wave.

Remark 1.1. Assumption (H2), corresponding to *evolutionarity* of (1.7) considered as an equation on the manifold of periodic solutions, is necessary for spectral stability in the sense usually defined; specifically, as described in Lemma 1.6, it is implied by condition (D3) below. Thus, there is no loss of generality, and considerable gain in clarity, in assuming (H2) from the outset as we do here.

Remark 1.2. Since our analyis is built to deal with nondegenerate cases, when we treat systems where bulk forces are incorporated in all the equations as in reaction-diffusion or convection-reaction-diffusion systems in nondivergence form, a similar count of dimensions leads us to assume

[6]The needed structural conditions may indeed be deduced from assumptions (H1)-(H3) and (D1)-(D3) below.

that, up to translation, the set of 1-periodic solutions forms a smooth 1-dimensional manifold

$$\{\ (U(k;\cdot), c(k))\ |\ k \in \Omega\ \} = \left\{\ (U^k(\cdot), c(k))\ \middle|\ k \in \Omega\ \right\}$$

where Ω is some open interval, with no constant of integration, hence no additional parameters $\mathcal{M}$ involved. The analog of (1.7) is then a scalar equation for the evolution of local wavenumber κ.

Three kinds of modulation are involved in (1.6): modulation in phase, wave number and mean. However, a prominent role is played by modulation in phase. Indeed, it is a familiar scenario that stability of patterns involve description of the evolution of various modulation parameters, and that among them the major role is devoted to parameters determining spatial positions. Note however that whereas for patterns whose variation is essentially localized in space, such as fronts, solitons or multi-solitons, kinks, shocks, etc., there are a finite number of parameters to follow, with their evolution described by a system of ordinary differential equations, here, in the periodic setting, there is a continuous description involving function-valued modulation parameters whose evolution obeys a partial differential system (here (1.7)), the reduction being not from continuous to discrete dynamics but from dynamics about periodic solutions to dynamics about constants: an averaging process. Note also that the special role of phase is already encoded in the formal description (1.6)-(1.7) since the parabolic nature of (1.7) hints at $(\mathcal{M}-\bar{M}, \kappa-\bar{k}) = (\mathcal{M}-\bar{M}, \bar{k}\partial_x(\Psi-\mathrm{Id})) << \Psi-\mathrm{Id}$.

This explains why, for the purposes of the stability analyses of [29, 30], it was sufficient to retain from (1.6)-(1.7) only the coarser approximation

$$u(x,t) \sim U^{(\bar{M},\bar{k})}(\Psi(x,t)) \tag{1.9}$$

neglecting all but phase modulations. Yet in doing so one gives up any hope to describe the precise behavior of the phase Ψ appearing in (1.9) since this would require a full modulation approximation and in particular knowledge of $\mathcal{M}$. Without this precise description of the phase, decay rates obtained for $\Psi - \mathrm{Id}$ and its derivatives may indeed seem mysterious.

In contrast, let us explain what may be guessed from (1.6)-(1.7) about behavior under localized perturbations, that is when initially $\kappa_0-\bar{k}$ is mean-free. To which extent this simplification will lead to higher order decay rates for $\Psi - \mathrm{Id}$ or $\partial_x(\Psi - \mathrm{Id})$ (directly needed to analyze (1.9)) is of course related to whether at some order the hyperbolic part of the equation for κ_t uncouples from the full system (1.7). In particular, if to second order ω is independent of $\mathcal{M}$ then one recovers for the phase higher decay rates corresponding to simpler systems for which there is no extra parameter $\mathcal{M}$ and (1.7) is reduced to a scalar conservation law, typical examples being reaction-diffusion systems treated in [25, 26, 57]; in other words decay rates for wave number perturbation are those of a solution of a viscous Burgers equation with mean-free initial datum. If, on the other hand, ω is independent of $\mathcal{M}$ only up to linear order, then intermediate decay rates are obtained,[7] slower than those for reaction-diffusion systems but faster than those for the general situation when no uncoupling is present or when the wave undergoes a nonlocalized perturbation.

The latter observations, to be established rigorously in the following, were not only inaccessible to proof but, as discussed in [29, "Discussion and open problems"], actually undecidable from the point of view of the lower-order description (1.9) of [29, 30]. Indeed, translated into the present terminology, the question posed in [29] which of Ψ, Ψ_x is the primary variable (with respect to true behavior) is essentially the question whether a localized initial perturbation induces nontrivial data for k in (1.7), the answer to which is a key step in our analysis, and a rather technical one.

With this is mind, our strategy will be first to validate the scenario (1.6)-(1.7), then to derive some consequences from the analysis of (1.7). But we first need to give precise definitions of terms such as "spectrally stable" ((D1)-(D3) below), "localized perturbation," and "linearly uncoupled."

[7]These intermediate decay rates require some assumptions about characteristics speeds provided by assumption (H3) below; see Remark 1.21.

1.2. **Setting and preliminary observations.** Linearizing (1.1) about $\bar{U}$ yields the periodic coefficient equation

$$(\partial_t - L)v = 0, \qquad Lv := (\bar{k}^2\partial_x^2 + \bar{k}\bar{c}\partial_x - \bar{k}\partial_x A)v, \qquad A(x) := df(\bar{U}(x)), \tag{1.10}$$

where here L is considered as a closed operator acting on $L^2(\mathbb{R};\mathbb{R}^n)$ with densely defined domain $H^2(\mathbb{R};\mathbb{R}^n)$.[8] Introducing the family of operator-valued symbols

$$L_\xi := e^{-i\xi\cdot}Le^{i\xi\cdot} = \bar{k}^2(\partial_x + i\xi)^2 + \bar{k}(\partial_x + i\xi)(\bar{c} - A), \quad \xi \in [-\pi, \pi],$$

operating on periodic functions on $[0,1]$, determined by the defining relation

$$L(e^{i\xi\cdot}f) = e^{i\xi\cdot}(L_\xi f) \quad \text{for } f \in H^2_{\text{per}}([0,1]), \tag{1.11}$$

we define following [59, 60, 29, 30, 33, 4] the *diffusive spectral stability* conditions:

(D1) $\sigma(L) \subset \{\lambda \mid \Re\lambda < 0\} \cup \{0\}$.
(D2) There exists a $\theta > 0$ such that for all $\xi \in [-\pi, \pi]$ we have $\sigma(L_\xi) \subset \{\lambda \mid \Re\lambda \le -\theta|\xi|^2\}$.
(D3) $\lambda = 0$ is an eigenvalue of L_0 with generalized eigenspace Σ_0 of dimension $n+1$.

Remark 1.3. As the coefficients of L are 1-periodic, Floquet theory implies that the spectrum of L considered as an operator on $L^2(\mathbb{R})$ is purely continuous, and that $\lambda \in \sigma(L)$ if and only if the spectral problem $Lv = \lambda v$ has an $L^\infty(\mathbb{R})$ eigenfunction of the form $v(x;\lambda,\xi) = e^{i\xi x}w(x;\lambda,\xi)$ for some $\xi \in [-\pi,\pi]$ and $w(\cdot;\lambda,\xi) \in L^2_{\text{per}}([0,1])$; that is, that

$$\sigma_{L^2(\mathbb{R})}(L) = \bigcup_{\xi\in[-\pi,\pi]} \sigma_{L^2_{\text{per}}([0,1])}(L_\xi).$$

See [19] for more details. In particular, since the spectrum of a given operator L_ξ is purely discrete, consisting of isolated eigenvalues of finite multiplicity which, furthermore, depend continuously on ξ, this provides a discrete parameterization of the essential spectrum of L.

Applying standard spectral perturbation theory [34] to the operators L_ξ, we obtain from (D3) that there exists for ξ sufficiently small an invariant $(n+1)$-dimensional subspace Σ_ξ of L_ξ and associated total eigenprojection $\Pi(\xi)$ bifurcating analytically from Σ_0 and its associated eigenprojection Π_0, with all other eigenvalues of L_ξ having real part uniformly bounded above by some negative constant[9].

Remark 1.4. Variations $\partial_M U_{|(\bar{M},\bar{k})}$, $\bar{U}'$ along the manifold of nearby periodic solutions lie always in Σ_0 (see proof of Lemma 1.5), accounting for $n+1$ dimensions, whereas $\partial_k U_{|(\bar{M},\bar{k})}$ usually does not. Thus, (D3) is an assumption of minimal dimension, corresponding also to the assumption that there are no neutral modes of L_0 other than those accounted for by modulation along the "slow manifold" $U^{(M,k)}(\,\cdot\, + \beta)$ as in (1.6). Assumption (D2) may be recognized as an "asymptotic parabolicity" assumption encoding time-asymptotic diffusion comparable to that of a second-order heat equation; it is directly related to the parabolicity of system (1.7). Assumption (D1) encodes that the spectrum corresponding to marginal stability is minimal, thus confined to $\{0\}$.

The following observation hints what may be gained at the linear level from "linear uncoupling" $\partial_M c|_(\bar{M},\bar{k}) \neq 0$, at the same time relating Σ_0 explicitly to variations along the manifold of periodic traveling waves nearby $\bar{U}$. Note [62], that this condition corresponds with decoupling of the κ equation in the first-order part of the linearization about $(\bar{M},\bar{k})$ of the Whitham system (1.7).

[8]Henceforth, in our notation for Lebesgue and Sobolev spaces we will suppress the definition of the range; in particular, we will write $L^p(\mathbb{R})$ for the equivalence class of p-integrable $\mathbb{R}^n$-valued functions $L^p(\mathbb{R};\mathbb{R}^n)$.

[9]Mark that the important property we have used through these arguments and obtained from the introduction of Bloch symbols L_ξ is compactness, which plays for this periodic setting the role of the finite dimensionality that one obtains with Fourier symbols associated to constant-coefficient operators.

Lemma 1.5. *Assuming (H1)–(H2) and (D3), L_0 has a nontrivial Jordan block at $\lambda = 0$ if and only if $\partial_M c \neq 0$ at $(\bar M, \bar k)$, or equivalently $\partial_M \omega(\bar M, \bar k) \neq 0$ in* (1.7)*, in which case there is a single Jordan chain of height two ascending from the genuine right eigenfunction $\bar U'$. In either case,*

$$\Sigma_0 = \mathrm{Span}\{\partial_M \bar U, \bar U'\}.$$

Proof. Variations $\partial_M \bar U$ comprise an n-dimensional subspace of solutions of

$$L_0 \partial_M \bar U = -\bar k(\partial_M c|_{(\bar M,\bar k)})\bar U',$$

i.e., either the eigenvalue or generalized eigenvalue equation at $\lambda = 0$, complementary to genuine eigenfunction $\bar U'$ (mean zero, hence independent of $\partial_M \bar U$ by $\int_0^1 \partial_M \bar U(x)dx = \mathrm{Id}$, a consequence of (1.5)). Comparing dimensions, we thus have $\Sigma_0 = \mathrm{Span}\{\partial_M \bar U, \bar U'\}$. If $\partial_M c|_{(\bar M,\bar k)} = 0$, then $\mathrm{Span}\{\partial_M \bar U\}$, hence also Σ_0, consists entirely of genuine eigenfunctions. If, on the other hand, there is a direction ν in which $\partial_M c|_{(\bar M,\bar k)} \cdot \nu \neq 0$, then there is a single generalized eigendirection $\mathrm{Span}\{\partial_M \bar U \cdot \nu\}$ over $\bar U'$. □

The following lemma justifies the apparently special assumption (H2).

Lemma 1.6. *Assuming (H1), (D3) implies (H2).*

Proof. Observe that variations with respect to (U_0, q) in (1.3) satisfy the eigenvalue ODE for L_0 while variations in c satisfy the generalized eigenvalue equation associated with genuine eigenfunction $\bar U'$, hence the subspace of all elements of their linear span satisfying the constraint of periodicity is contained in Σ_0. Condition (D3) implies that this subspace is dimension $\leq n+1$, from which we may deduce that the n-dimensional periodicity condition $U(1) = U_0$ is full rank at the values $(\bar k, \bar c, \bar U_0, \bar q)$ corresponding to $\bar U$, as the kernel with respect to the $2n+1$ parameters (c, U_0, q) is dimension $\leq n+1$. This guarantees existence of a smooth parametrization $U(\alpha, \beta; \cdot) = U^\alpha(\cdot + \beta)$, with (α, β) lying in some open set of $\mathbb{R}^n \times \mathbb{R}$, of the manifold of nearby periodic solutions, whereupon we may conclude using (D3) again that $\det \partial_\alpha (M, k)_{|(\bar\alpha,\bar\beta)} \neq 0$ by [52, Theorem 1.3].[10] Thus $(\alpha, \beta) \mapsto (M, k, \beta)$ is locally invertible, yielding a smooth parametrization by (M, k, β). □

We complete our set of assumptions with a final nondegeneracy condition corresponding to *strict hyperbolicity* at $(\bar M, \bar k)$ of the first-order part of (1.7), namely, the assumption:

(H3) the eigenvalues a_j of $\frac{\partial(F-\bar c M, -\omega - \bar c k)}{\partial(M,k)}|_{(\bar M,\bar k)}$ are distinct.

The role of this assumption is made clearer by the following connections established at the linear spectral level in [62, 52] between the Whitham system (1.7) and long-time ($\sim$ low-frequency for parameters, $\sim$ low-Floquet exponent ξ for original functions) behavior.

Proposition 1.7 ([50, 62])**.** *Assuming (H1)-(H2), the $(n+1)$-multiplicity eigenvalue $\lambda = 0$ of L_ξ at $\xi = 0$ bifurcates in a differentiable way for $\xi \neq 0$ sufficiently small into $n+1$ eigenvalues*

$$\lambda_j(\xi) = -i\bar k \xi a_j + o(\xi), \quad j = 1, \dots, n+1, \tag{1.12}$$

where a_j are the eigenvalues of $\partial_{(M,k)}(F - \bar c M, -\omega - \bar c k)|_{(\bar M,\bar k)}$, that is, $\bar k a_j$ are the characteristic velocities of the first-order part of the Whitham modulation equations (1.7) *at the values $(\bar M, \bar k)$ associated with $\bar U$. Moreover, assuming (H1)–(H3), this bifurcation is analytic.*

Proposition 1.7 was established in [62, 52] using direct Evans function calculations. In Section 3.1, we provide an alternative proof based on direct spectral perturbation expansion (as in [48, 49]) which is better suited to the techniques utilized in our analysis, and yields also information about eigenprojections. In the meantime, we observe the following interesting corollary.

[10] A periodic Evans function computation showing that the $(n+1)$st derivative of the Evans function at $\lambda = 0$ is proportional to $\det \partial_\alpha (M, k)$, hence $\det \partial_\alpha (M, k)_{|(\bar\alpha,\bar\beta)} \neq 0$ is necessary for (D3) (implicit also in the earlier work [62]).

Corollary 1.8. *Assuming (H1)-(H2), $\sigma(L) \subset \{\lambda \,|\, \Re\lambda \leq 0\}$ implies that characteristics a_j are real. That is, weak hyperbolicity (a_j real)*[11] *of the first-order Whitham equations at $(\bar{M}, \bar{k})$ is necessary for spectral stability; in particular both (D1) and (D2) imply this notion of weak hyperbolicity.*

Corollary 1.8 gives rigorous validation of the Whitham equations as formal predictors of stability. Indeed, their hyperbolicity is often used as a definition of "modulational stability."

Remark 1.9. Assumption (H3) provides two kinds of regularity in a simple unified way: on one hand it gives the analyticity of critical spectral modes of L, and on the other hand, when combined with weak hyperbolicity (here following from (D1) or (D2)), it yields strict hyperbolicity of the Whitham's system. We expect that, by usual considerations, it could be replaced with symmetrizability of the Whitham's system and a direct smoothness assumption on spectral expansions.

We still need to say some words about what we mean by a nonlocalized perturbation. First, a localized perturbation of $\bar{U}$ is something that may be written as $\bar{U} + v$ with v localized (and smooth), say $v \in L^1(\mathbb{R}) \cap H^K(\mathbb{R})$. Note that in order for the process of gluing together a left portion of the original wave, some function on a finite interval, and a right portion of the original wave to yield a localized perturbation (according to our definition), the left-hand and right-hand copies of the original wave should be in phase. It is this stringent condition (corresponding to a mean-free condition for the local wave number) that we want to relax in going to nonlocalized perturbations.

Thus, rather than localized perturbations, we consider perturbations of the type $(\bar{U}+v)\circ\Psi$ with v and $\partial_x(\Psi - \mathrm{Id})$ localized, allowing for changes in phases between limiting left and right waves ($\Psi - \mathrm{Id}$ is not localized) but not a change in the waves themselves, for instance in its wave number or its mean. In other words, our nonlocalized perturbations will still yield localized data for the Whitham system (1.7).

1.3. **Results and implications.** With these preparations, we are ready to state our two main theorems. The first one is an extension of [29, 30] to stability under nonlocalized perturbations. The second one provides asymptotic behavior by validating the scenario (1.6)-(1.7).

Here, and throughout the paper, given two real valued functions A and B, we say that $A \lesssim B$ or that for every $x \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$, $A(x) \lesssim B(x)$ if there exists a constant $C > 0$ such that $A(x) \leq CB(x)$ for each $x \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$. Even in a chain of inequalities, we will also feel free to denote by C harmless constants with different values.

Theorem 1.10 (Stability)**.** *Let $K \geq 3$. Assuming (H1)–(H3) and (D1)-(D3), let*

$$E_0 := \|\tilde{u}_0(\cdot - h_0(\cdot)) - \bar{U}(\cdot)\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})} + \|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})}$$

be sufficiently small, for some choice of phase shift h_0. Then, there exists a global solution $\tilde{u}(x,t)$ of (1.1) *with initial data $\tilde{u}_0$ and a phase function $\psi(x,t)$ such that $\psi(\cdot,0) = h_0$ and, introducing a global phase shift $\psi_\infty = (h_0(-\infty) + h_0(\infty))/2$, for $t \geq 0$ and $2 \leq p \leq \infty$,*

$$\begin{aligned} \|\tilde{u}(\cdot - \psi(\cdot,t),t) - \bar{U}(\,\cdot\,)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)} \\ \|\nabla_{x,t}\,\psi(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)}, \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} \|\tilde{u}(t) - \bar{U}(\,\cdot\, - \psi_\infty)\|_{L^\infty(\mathbb{R})}, \quad \|\psi(t) - \psi_\infty\|_{L^\infty(\mathbb{R})} &\lesssim E_0, \\ \|\tilde{u}(t) - \bar{U}\|_{L^\infty(\mathbb{R})} &\lesssim E_0 + |\psi_\infty \textit{ mod } 1|; \end{aligned} \tag{1.14}$$

[11]Full hyperbolicity requiring of course also semisimplicity of a_j as eigenvalues of $\frac{\partial(F - \bar{c}M, -\omega - \bar{c}k)}{\partial(M,k)}|_{(\bar{M},\bar{k})}$.

in particular

$$\|\tilde u(\cdot,t)-\bar U(\cdot+\psi(\cdot,t))\|_{L^p(\mathbb{R})}\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)}. \tag{1.15}$$

Remark 1.11. The above result suggests the introduction of "space-modulated distances"[12]

$$\delta_X(u,v)\ =\ \inf_{\Psi}\quad \|u\circ\Psi-v\|_X\ +\ \|\partial_x(\Psi-\mathrm{Id})\|_X.$$

In these terms, it states $\delta_{L^1\cap H^K}-\delta_{L^2\cap L^\infty}$ asymptotic stability,[13] and $\delta_{L^1\cap H^K}-\|\cdot\|_{L^\infty}$ bounded (orbital) stability. In the following, among other things, we discuss situations, involving appropriate uncoupling conditions, under which one may go from this "space-modulated" asymptotic stability to the usual $\|\cdot\|_{L^1\cap H^K}-\|\cdot\|_{L^2\cap L^\infty}$ asymptotic stability. Note that this notion of "space-modulated" stability is a natural generalization of the more common one of orbital stability for patterns with localized variations (e.g. fronts, shocks, kinks, solitons, etc.), where the above infimum is taken over uniform translations only.

Theorem 1.12 (Asymptotic behavior). *Let $\eta>0$, arbitrary, and $K\geq 4$. Under the assumptions of Theorem 1.10, and suitable parametrization there exist $M(x,t)$, and $\psi(x,t)$ such that $\psi(\cdot,0)=h_0$ and, with global phase shift $\psi_\infty=(h_0(-\infty)+h_0(\infty))/2$, for $t\geq 0$, $2\leq p\leq\infty$,*

$$\begin{aligned}\|\tilde u(\cdot-\psi(\cdot,t),t)-U^{\bar M+M(\cdot,t),\bar k/(1-\psi_x(\cdot,t))}(\cdot)\|_{L^p(\mathbb{R})}&\lesssim E_0\ \ln(2+t)\,(1+t)^{-\frac{3}{4}},\\ \|(\bar k\psi_x,M)(t)\|_{L^p(\mathbb{R})}&\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)},\\ \|\psi(t)-\psi_\infty\|_{L^\infty(\mathbb{R})}&\lesssim E_0.\end{aligned} \tag{1.16}$$

Moreover, setting $\Psi(\cdot,t)=(\mathrm{Id}-\psi(\cdot,t))^{-1}$, $\kappa=\bar k\partial_x\Psi$, $\mathcal{M}(\cdot,t)=(\bar M+M(\cdot,t))\circ\Psi(\cdot,t)$, and defining $(\mathcal{M}_W,\kappa_W)$ and Ψ_W to be solutions of equations (1.7)*, and* (1.8) *with initial data*

$$\begin{aligned}\mathcal{M}_W(\cdot,0)&=\bar M+\tilde u_0-\bar U\circ\Psi(\cdot,0)+\left(\frac{1}{\partial_x\Psi(\cdot,0)}-1\right)\left(\bar U\circ\Psi(\cdot,0)-\bar M\right),\\ \kappa_W(\cdot,0)&=\bar k\partial_x\Psi(\cdot,0),\\ \Psi_W(\cdot,0)&=\Psi(\cdot,0),\end{aligned} \tag{1.17}$$

we have, for $t\geq 0$, $2\leq p\leq\infty$,

$$\begin{aligned}\|(\mathcal{M},\kappa)(t)-(\mathcal{M}_W,\kappa_W)(t)\|_{L^p(\mathbb{R})}&\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta},\\ \|\Psi(t)-\Psi_W(t)\|_{L^p(\mathbb{R})}&\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)+\eta};\end{aligned} \tag{1.18}$$

in particular, $\kappa=\bar k\partial_x\Psi$, $\kappa_W=\bar k\partial_x\Psi_W$, and

$$\begin{aligned}\|\tilde u(\cdot,t)-U^{\mathcal{M}(\cdot,t),\kappa(\cdot,t)}(\Psi(\cdot,t))\|_{L^p(\mathbb{R})}&\lesssim E_0\ \ln(2+t)\,(1+t)^{-\frac{3}{4}},\\ \|\tilde u(\cdot,t)-U^{\mathcal{M}_W(\cdot,t),\kappa_W(\cdot,t)}(\Psi_W(\cdot,t))\|_{L^p(\mathbb{R})}&\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)+\eta}.\end{aligned} \tag{1.19}$$

Remark 1.13. A suitable choice in parametrization is made here to ensure that interdependences on k and β in $U^{(M,k)}(\cdot+\beta)$ are compatible with the expected relation between local phase and local wave number, $\kappa=\bar k\partial_x\Psi$. Explicitly, our normalizing choice is performed in (4.2) (which involves $\tilde q_{n+1}(0)=\bar u^{adj}$ defined in Proposition 3.1) to get Lemma 4.1.

Remark 1.14. Prescription of the initial data (1.17), especially for $\mathcal{M}_W(\cdot,0)$, is a subtle point[14] not evident from the viewpoint of formal approximation (1.6)-(1.7). In particular, the appearance of a term related to phase variations in (1.17)(i) arises in our analysis through a detailed study of the contribution of high frequencies of the local wave number to variations of the low Floquet number part

[12]These are not true distances, but rather measures associated with a seminorm.

[13]The proof gives also a $\delta_{L^1\cap H^K}-\delta_{H^K}$ asymptotic stability.

[14]This issue does not arise in the related analysis [26] of the reaction-diffusion case, as M does not appear.

of the solution (see the key equality (3.27)). Nevertheless, in the end, each term involved in (1.17)(i) has a nice interpretation, with $\tilde{u}_0 - \bar{U} \circ \Psi(\cdot, 0)$ accounting for the contribution of amplitude variations to the initial perturbation of the mean $\bar{M}$ and $(1/\partial_x \Psi(\cdot, 0) - 1)\left(\bar{U} \circ \Psi(\cdot, 0) - \bar{M}\right)$ encoding the contribution of period variations. To be more specific, on one hand, setting $\tilde{d}_0 = \tilde{u}_0 - \bar{U} \circ \Psi(\cdot, 0)$, we observe that $\tilde{d}_0$ differs from $x \longmapsto \int_{-1/2}^{1/2} \tilde{d}_0(x+y)\, dy$, which is easier to interpret, by a localized zero-mean function, a difference that is asymptotically irrelevant[15] at our level of description. Likewise, assuming on the other hand validity of the approximation $\bar{U} \circ \Psi(\cdot, 0) \sim \bar{U} + (\Psi(\cdot, 0) - \mathrm{Id}_{\mathbb{R}})\bar{U}'$ leads us to consider $x \longmapsto \int_{-1/2}^{1/2} (\Psi(x+y, 0) - (x+y))\bar{U}'(x+y)\, dy$ which is

$$x \longmapsto -\int_{-\frac{1}{2}}^{\frac{1}{2}} (\partial_x \Psi(x+y, 0) - 1)\bar{U}(x+y)\, dy \ + \ \bar{U}\big(x - 1/2\big)\left[\Psi\big(x + 1/2, 0\big) - \Psi\big(x - 1/2, 0\big)\right]$$

where, up to localized zero-mean functions, the first part of the sum is $-(\partial_x \Psi(\cdot, 0) - 1)\bar{U}$ and the second reduces to $(\partial_x \Psi(\cdot, 0) - 1)\bar{M}$. Note that the fact that in the end of this latter formal computation we recover the formula of the Theorem only in an approximate way reveals that the first approximation in the argument is invalid. Yet this incorrect approximation possesses a correct analog (see Section 2.2) leading to the initial data of the Theorem. Note also that any small localized perturbation of $(\bar{M}, \bar{k})$ may be realized as initial data in (1.17) by appropriately choosing h_0 and $\tilde{u}_0$ so that at our level of accuracy the full dynamics of (1.7) near $(\bar{M}, \bar{k})$ are present in (1.1) around $\bar{U}$.

Remark 1.15. Bounds (1.19) are both of form (1.6), with $\bar{k}\Psi_x = \kappa$, validating a slow modulation picture of behavior. Yet comparison of (1.19)(i) with (1.19)(ii) reveals that by allowing $(\mathcal{M}, \kappa)$ to satisfy (1.7) only in an approximate way we here construct a phase modulation more accurate at least by factor $(1+t)^{-1/4-\eta}$ than that of the formal Whitham construction (1.6)-(1.7).[16] This is again a manifestation of the fact that comparisons to periodic functions are very sensitive even to small perturbations in description of respective spatial positions as encoded by local phases.

Decay for localized perturbations. Standard bounds on localized solutions of systems of parabolic conservation laws of form (1.7) (see Proposition B.1 below) show that in general

$$\|\kappa(t) - \bar{k}\|_{L^p(\mathbb{R})} \sim (1+t)^{-\frac{1}{2}(1-1/p)}, \quad \|\Psi(t) - \mathrm{Id}\|_{L^\infty(\mathbb{R})} \sim 1,$$

so that

$$\|U^{\mathcal{M}(\cdot,t),\kappa(\cdot,t)}(\Psi(\cdot, t)) - \bar{U}(\cdot)\|_{L^\infty(\mathbb{R})} \sim \|\Psi(t) - \mathrm{Id}\|_{L^\infty(\mathbb{R})} \sim 1 \tag{1.20}$$

$\gg (1+t)^{-\frac{1}{2}+\eta}$. Together with (1.19), this rigorously validates the formal Whitham approximation while simultaneously showing that estimates (1.13)–(1.14) are sharp for nonlocalized perturbations, $h_0 \not\equiv 0$, leading always to nontrivial localized data (1.17) in κ_W for the Whitham system (1.7), and for localized perturbations $h_0 \equiv 0$ are sharp in the generic case where no uncoupling is present.

However, an interesting further implication of (1.19) is that when κ_W decouples to sufficient order from the rest of the Whitham equations, the estimates (1.13)–(1.14) *can be sharpened for localized perturbations, to yield asymptotic decay.* To make this latter point precise, we introduce the following definitions.

Definition 1.16. We say that a wave is *linearly phase-decoupled* if $\partial_M c|_{(\bar{M},\bar{k})} = 0$, or, equivalently, $\partial_M \omega(\bar{M}, \bar{k}) = 0$ in (1.7): that is, κ is a characteristic variable for (1.7) at the special point $(\bar{M}, \bar{k})$. Otherwise, we will say that it is *linearly phase-coupled*, or simply "generic type."

[15]This follows from the general theory for parabolic systems of conservation laws, see for instance Proposition B.1.

[16]Here we are using the additional fact (not explicitly stated here) that estimate (1.19)(ii) is sharp. On the other hand, we do not expect (1.19)(i) to be sharp (see Remark 2.1).

Definition 1.17. We say that a wave is *quadratically phase-decoupled* if both $\partial_M c|_{(\bar M,\bar k)} = 0$ and $\partial_M^2 c|_{(\bar M,\bar k)} = 0$. or, equivalently, $\partial_M \omega(\bar M,\bar k) = 0$ and $\partial_M^2 \omega(\bar M,\bar k) = 0$ in (1.7).

This simple classification unifies and generalizes a number of observations in [50, 51, 29, 30]. As we have seen in Lemma 1.5 and shall discuss further in Remark 3.2, linear phase-decoupling implies that *to linear order* the phase behaves similarly as in the reaction-diffusion case studied in [59, 60, 28] (localized perturbations) and [25, 26, 57] (nonlocalized), for which the associated Whitham system consists of a single equation[17]

$$\kappa_t - \bar k(\omega(\kappa) + \bar c\kappa)_x = \bar k^2 (d(\kappa)\kappa_x)_x, \qquad \omega(k) = -k\, c(k),$$

encoding the nonlinear dispersion relation induced by the periodic existence theory. In particular, we shall show that spectrally stable linearly phase-decoupled waves like spectrally stable reaction-diffusion waves are *linearly and nonlinearly asymptotically stable* and not only boundedly stable *with respect to localized perturbations.* Yet *at the nonlinear level* the phase behaves similarly as in the reaction-diffusion case, sharing the same decay rates, only if quadratic decoupling is present. The situation is actually simpler in higher dimensions where the asymptotic dynamics are essentially linear and the distinction between linearly decoupled and generic cases is sufficient [53, 29]. We make these observations precise in the following corollary. For a proof, see Appendix B.2.2.

Corollary 1.18 (Localized perturbations)**.** *Under the assumptions of Theorem 1.12, for localized perturbations $h_0 \equiv 0$, and ψ defined as in Theorem 1.12, if $\bar U$ is linearly phase-decoupled and $E_1 := E_0 + \||\cdot|\,(\tilde u_0 - \bar U)\|_{L^1(\mathbb{R})}$ is sufficiently small, then, for $t>0$ and $2 \le p \le \infty$,*

$$\begin{aligned} \|\nabla_{x,t}\psi(t)\|_{L^p(\mathbb{R})} &\lesssim E_1(1+t)^{-\frac12(1-1/p)-\frac14+\eta}, \\ \|\tilde u(t) - \bar U\|_{L^p(\mathbb{R})}, \quad \|\psi(t)\|_{L^p(\mathbb{R})} &\lesssim E_1(1+t)^{\frac{1}{2p}-\frac14+\eta}, \end{aligned} \tag{1.21}$$

while if $\bar U$ is quadratically phase-decoupled and E_0 is sufficiently small, then

$$\begin{aligned} \|\nabla_{x,t}\psi(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac12(1-1/p)-\frac12+\eta}, \\ \|\tilde u(t) - \bar U\|_{L^p(\mathbb{R})}, \quad \|\psi(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac12(1-1/p)+\eta}\ ; \end{aligned} \tag{1.22}$$

for $t>0$ and $2 \le p \le \infty$. In either case, $\bar U$ is nonlinearly asymptotically stable *from $L^1(\mathbb{R};(1+|x|)dx) \cap H^K(\mathbb{R})$ to $L^p(\mathbb{R})$, for all $2 < p \le \infty$.*

Remark 1.19. Comparing bounds (1.22) for localized perturbations and (1.13)–(1.14) for nonlocalized perturbations to those obtained in [57, 25, 26], we see that bounds for the quadratically phase-decoupled case exactly match the bounds for reaction-diffusion systems.

Remark 1.20. One may wish to express localization as a mean-free condition on $\partial_x h_0$. Actually, in the above bounds, the condition $h_0 \equiv 0$ may indeed be relaxed to the condition that $\partial_x h_0$ is mean-free and either $E_1 := E_0 + \||\cdot|\,\partial_x h_0\|_{L^1(\mathbb{R})}$ is small in the quadratically phase-decoupled case or $E_1 := E_0\||\cdot|\,\partial_x h_0\|_{L^1(\mathbb{R})} + \||\cdot|\,(\tilde u_0(\cdot - h_0(\cdot)) - \bar U)\|_{L^1(\mathbb{R})}$ is small in the linearly phase-decoupled case. In either case, the conclusion is (asymptotic) orbital stability with asymptotic phase $\psi_\infty = (h_0(-\infty) + h_0(\infty))/2$ (in the sense of [21]).

Remark 1.21. For the analysis of localized perturbations in the linearly phase-decoupled case, assumption (H3) plays a role deeper than just providing regularity in a simple way. Indeed, in this case, the extra damping $(1+t)^{-1/4}$ in (1.21) encodes the fact that quadratic interactions between diffusion waves traveling at different characteristic speeds are asymptotically irrelevant ([40], see Remark B.4). Thus, here one should not expect to be able to replace (H3) with something weaker

[17] See Remark 1.2.

than: the linear group velocity associated to the wavenumber mode is different from all other characteristic speeds.

We emphasize again that Corollary 1.18 follows from the validation of (1.6)-(1.7) and the known theory for behavior, about constant states, of solutions to hyperbolic-parabolic systems. Indeed linear uncoupling implies that only one of the hyperbolic modes contributes significantly to phase variations. This implies that when considering localized perturbations for the original system, we should put essentially a constant as initial data along this caracteristic mode and thus the coordinate along this mode is slaved to the evolution of the other modes through nonlinear interactions. In the strictly hyperbolic case that we consider here and when no further uncoupling is present, this leads to a time decay enhanced by $(1+t)^{-1/4}$, corresponding to eventual mode-by-mode separation. Moreover, when there is actually no significant nonlinear contribution of the other modes (quadratic uncoupling), then there is an extra damping up to order $(1+t)^{-1/2}$, the level at which the (interacting) diffusion-waves scenario may be validated.

1.4. **Examples.** Having established the importance for asymptotic behavior of the Whitham equations, we now give some examples indicating their range of possible behaviors. In this section, we relax the restriction, made for expositional simplicity, to second-order parabolic semilinear systems of conservation laws and discuss a full range of models arising in applications, including reaction-diffusion equations (Example 1.22), equations with higher-order or partial diffusion (Examples 1.22, 1.25 and 1.26), and even mixed conservative/nonconservative equations (Example 1.26).

From the point of view of the present paper, the main example is Example 1.23, which illustrates for second-order parabolic semilinear conservation laws both phase-decoupling and phase-coupling. However, we emphasize that the analysis of all of these models may be carried out with minor changes within the same basic analytical framework set out here and in [29, 30]. We discuss this further in Appendix D, along with the question of numerical or analytical verification of the stability conditions, needed to conclude validity of the Whitham equations.

To simplify the discussion, we restrict to the first-order part of the Whitham equations, which suffices to determine the main qualitative features of solutions- in particular, phase-decoupling vs. coupling- and has a common derivation/form [62] independent of second- and higher-order terms.

Example 1.22 ([60])**.** The Swift-Hohenberg equation

$$u_t + (1+\partial_x^2)^2 u - ru + f(u) = 0, \tag{1.23}$$

where $r \in \mathbb{R}$ is a bifurcation parameter and f is some sufficiently smooth nonlinearity, admits for certain values of r periodic waves of speed $c \equiv 0$. This equation arises as a simplified equation for the Taylor-Couette problem and is proved to possess diffusively spectrally stable waves [60]. As a reaction-diffusion equation, with no conservative part, this yields (see, e.g., [24, 15, 57, 29, 30]) a scalar first-order Whitham equation[18]

$$\kappa_t = 0.$$

Example 1.23 ([50, 62, 7])**.** The equations of one-dimensional viscoelasticity with artificial viscosity and strain-gradient effects ("capillarity") may be expressed in Lagrangian coordinates, after a change of variables [58, 51], as

$$\begin{aligned} \tau_t - u_x &= \varepsilon_1 \tau_{xx}, \\ u_t - \sigma(\tau)_x &= \varepsilon_2 u_{xx}, \end{aligned} \tag{1.24}$$

where $\tau = \chi_x \in \mathbb{R}^d$ and $u = \chi_t \in \mathbb{R}^d$ are derivatives of deformation $\chi : \mathbb{R} \to \mathbb{R}^d$, σ is the stress-strain relation of the elastic material, and $\varepsilon_1, \varepsilon_2 > 0$ are scalar coefficients related to viscosity/capillarity; see [7]. It is readily verified by energy considerations [62, 50, 55] that periodic waves may only have

[18]See Remark 1.2.

speed $c = 0$. By this, together with Galilean invariance with respect to shifts in u, we find, setting T, U, Σ to be the means of $\bar\tau$, $\bar u$, and $\sigma(\bar\tau)$ over one period, that the associated first-order Whitham system has form

$$\begin{aligned} T_t - U_x &= 0,\\ U_t - \Sigma(T,\kappa)_x &= 0,\\ \kappa_t &= 0, \end{aligned} \tag{1.25}$$

hence, since $\omega(T,U,k) \equiv 0$ is evidently independent of (T,U), is quadratically (indeed, totally) phase-decoupled. That is, in terms of the phase equation principally determining behavior, (1.24) and (1.23) exhibit parallel behavior $\kappa_t = 0$,[19] despite their different origins. For localized data $h_0 \equiv 0$, giving $\kappa_0 \equiv \bar k$, this reduces to the first-order wave equation $T_t - U_x = 0$, $U_t - \Sigma(T,\bar k)_x = 0$, which is strictly hyperbolic when Σ is monotone decreasing, as occurs for some but not all cases [69]. By Corollary 1.18, any spectrally stable periodic solutions of (1.24) would be nonlinearly asymptotically stable with respect to localized perturbations, answering a question posed in [51, 62, 29].

The same equations written in Eulerian coordinates (in terms of $\rho := \tau^{-1}$ and $m := u/\tau$) are *phase-coupled* [62], hence at best only nonlinearly bounded stable, with the explanation that modulations in wave speed in this case lead to deviation of characteristic paths, hence solutions are no longer compared along the Lagrangian trajectories where they are most closely matched. To put things another way, coordinatization by Lagrangian markers accomplishes a substantial part of the modulation that in Eulerian coordinates yields decay estimate (1.13)(i).

Remark 1.24. It has been shown in [50, 55] that under a wide variety of circumstances, in particular, always for one-dimensional deformations, $d = 1$, periodic solutions of (1.24) are spectrally unstable, *whether or not the first-order Whitham equations* (1.25) *are of hyperbolic type.* This shows the importance of the full diffusive stability conditions, beyond the intuitive conditions of Corollary 1.8. It is an interesting open problem whether there exist stable waves for $d > 1$ [69, 55].

Example 1.25. For the Kuramoto–Sivashinsky equation

$$u_t + (u^2/2)_x + u_{xx} + u_{xxxx} = 0, \tag{1.26}$$

setting U, Σ to be the means of $\bar u$, and $\bar u^2/2$ over one period, and making use of the Galilean invariance $x \to x - ct$, $u \to u + c$, we find that $c(U_1 + U_2, k) = c(U_1, k) + U_2$ and $\Sigma(U_1 + U_2, k) = \Sigma(U_1, k) + U_1U_2 + U_2^2/2$. It is known that, within a certain parameter range, (1.26) supports odd- hence mean-free- profiles with $c \equiv 0$ (see Remark 1.27). For these solutions, $c(0,k) = 0$, hence $c(U,k) = U$. Thus, the associated first-order Whitham system is *linearly phase-coupled*, of form

$$\begin{aligned} U_t + \big(\Sigma(U_\star,\kappa) + (U - U_\star)^2/2\big)_x &= 0,\\ \kappa_t + ((U - U_\star)\kappa)_x &= 0\ . \end{aligned} \tag{1.27}$$

Linearizing about constant solution $(U_\star, \bar k)$ and reintroducing the phase through $k = \psi_x$, this gives a second-order wave equation $\psi_{tt} + \bar k(\partial_k\Sigma)(U_\star,\bar k)\ \psi_{xx} = 0$ in the phase [18] provided $(\partial_k\Sigma)(U_\star,\bar k) < 0$. As illustrated numerically in [5], linear phase-coupling has the effect that nonlocalized perturbations in the phase can arise even through localized initial perturbations. Numerical studies [18, 5] indicate that there exist "bands" in parameter space of spectrally stable waves, satisfying hypotheses (H1)–(H3), (D1)–(D3); see Appendix D for further discussion.

Example 1.26. The Saint-Venant equations for inclined thin-film flow appear in Lagrangian coordinates as

$$\begin{aligned} \tau_t - u_x &= 0,\\ u_t + ((2F)^{-1}\tau^{-2})_x &= 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x, \end{aligned} \tag{1.28}$$

[19]At second order, a linear heat equation $\kappa_t = d\kappa_{xx}$, $d > 0$.

where τ is the reciprocal of fluid height, u is velocity averaged with respect to depth, x denotes a Lagrangian marker moving with the flow, and ν and F are dimensionless constants, with force term $1-\tau u^2$ representing the balance between gravity and turbulent bottom friction. In terms of structure, this is intermediate between the reaction-diffusion case of (1.22) and (1.24), having a first equation in conservative (divergence) form and a second equation in nonconservative convection-reaction-diffusion form. The same derivation as for (1.7) yields the first-order Whitham system

$$\begin{aligned} T_t - U(T,\kappa)_x &= 0, \\ \kappa_t - (c(T,\kappa)\kappa)_x &= 0, \end{aligned} \tag{1.29}$$

where T and U are defined as the means of $\bar\tau$ and $\bar u$ over one period, and (T,k) parametrize the associated two-parameter family of periodic traveling waves with speed $c=c(T,k)$; see [33, 3, 49] for further details. For this model, the speed c is never zero, and in particular depends typically nontrivially on T. Thus, this system, like (1.27), is in general fully phase-coupled. Numerical experiments [3, 6] indicate that (1.29) can be either hyperbolic (consistent with stability) or elliptic (implying instability), depending on parameter values; moreover, there exists a band of parameters on which waves satisfy the stability hypotheses (H1)–(H3), (D1)–(D3).

Remark 1.27 (Phase-decoupling and symmetry). As illustrated by Example 1.23, phase-decoupling is not always an isolated degeneracy on a special set of parameters, but for models with special structure may hold on an open set of parameters/waves; indeed, more, we may have $c\equiv 0$. This is reminiscent of the well-known principle in the reaction-diffusion setting that, by reflection symmetry of the equations $u_t+f(u)=u_{xx}$, even-symmetric standing-wave solutions generically persist as families of solutions with $c(k)\equiv 0$. For, otherwise, the fact that reflection preserves k would violate local uniqueness of solutions as a function of k. Alternatively, one may observe that zero-speed waves satisfy a Hamiltonian ODE, hence, by a dimensional count, exhaust the available dimensions in the set of nearby solutions. This principle is illustrated in the behavior cited in Example 1.22.

Likewise, in Example 1.23, one finds [33, 3] that zero-speed waves satisfy a Hamiltonian ODE identical to that of the reaction-diffusion case, with d free parameters given by a constant of integration, hence, by a dimensional count, generically fill up the $(d+1)$-dimensional set of nearby solutions, giving $c(T,k)\equiv 0$. Similarly, the Kuramoto–Sivashinsky equations (1.26), are invariant under $x\to -x$, $c\to -c$, $u\to -u$, from which we may deduce that odd-symmetric zero-speed solutions generically persist, as cited in Example 1.25. For, otherwise, the fact that reflection preserves k would violate uniqueness with respect to k of solutions with fixed zero mean.

1.5. **Discussion and open problems.** Our results extend to the conservative case the results established recently for reaction-diffusion systems in [57, 25, 26] regarding behavior, and extend to nonlocalized perturbations the results obtained for conservation laws in [28, 29, 33, 3, 4] regarding stability under localized perturbation. The method of analysis used here is similar to but much more complicated than the arguments used in [25, 26] to study the reaction-diffusion case and the reader is encouraged to consult these references as motivation in a simpler context. As noted above, the methods of [57] do not seem to apply.

The main new difficulties overcome in the present analysis beyond that of [25, 26] are the treatment of nonlocalized perturbations in a way including the phase-coupled case, which has an essentially different Jordan block structure from that of the phase-decoupled case, and the identification of the Whitham equations with the asymptotic second-order modulation system arising naturally in our analysis via a system of integral equations. The latter task involves surprisingly subtle aspects not present in the reaction-diffusion case concerning the influence of phase modulation Ψ on the mean $\mathcal{M}$, first, through high-frequency resonances, on its initial data (see Remark 1.14), and, second, through the influence of the implicit nonlinear change of independent coordinates (2.5) used in our nonlinear iteration scheme on the form of the Whitham equations (see Section B.3).[20] Though we give a unified proof, regardless of coupling distinctions, mainly out of a desire for clarification of the essential features of modulation theory, a proof of Theorem 1.12 would not be much simpler had we restricted it to the linearly-uncoupled case.

Comparing bounds (1.16)-(1.19) to the corresponding bounds for reaction-diffusion systems in [26, Theorem 1.3], we see that they are identical; that is, modulations are equally well-approximated for systems of conservation laws as for reaction-diffusion systems by the formal Whitham approximation (1.7). It follows (through Proposition B.1) that for nonlocalized perturbations, *behavior and decay rates are also essentially identical in these two cases,* as the formal asymptotics suggest.

On the other hand, for localized data, $h_0 \equiv 0$, the decay estimates established for the reaction-diffusion case in [53, 28] are faster by a factor $(1+t)^{-\frac{1}{2}}$, or "roughly one derivative" in terms of standard heat bounds, than those of Theorem 1.10 in the generic conservative case, which are the same for localized as for nonlocalized perturbations. As discussed in [30], linearly phase-decoupled waves exhibit a similar behavior at the linearized level. However Corollary 1.18 shows that a quadratic decoupling is needed to yield a similar behavior at the nonlinear level, while waves that are linearly phase-decoupled show an intermediate behavior, asymptotic stability but with slower rates.

At broadest level, our results confirm that an accurate distinction is not between reaction-diffusion and conservation law systems but between *phase-decoupled*, and *non-phase-decoupled* waves (the former trivially including the reaction-diffusion case), which indeed exhibit the asymptotic behavior suggested by their common formal asymptotic description in terms of the Whitham equation(s). A key new piece of information supplied by our analysis that is not present in the formal Whitham derivation is the way in which initial data is taken on by the time-asymptotic Whitham system. Though the ultimate prescription in Theorem 1.12 is simple, it is determined by a detailed series of linear and nonlinear estimates that are quite far from the techniques of formal asymptotic expansion.

We stress, finally, that our nonlinear iteration scheme is quite robust. In particular, there is no use of analytic semi-group properties in our argument, hence it is not sensitive to changes in order (e.g., to KS or KS-KdV) or type (e.g., quasilinear or degerate as for Saint-Venant) of the equations under study. To control regularity, we mainly use nonlinear H^s damping estimates and C^0 semi-group resolvent bounds in H^s; both given by standard energy estimates techniques (Kawashima's if needed [35, 72, 74]). This allows a wide range of generalizations, as discussed in Appendix D.

The diffusive spectral stability conditions (D1)–(D3) have been shown numerically to hold for "bands" of stable periodic waves, in several interesting settings, and with a high degree of precision; see, for example, [18, 4, 5]. We view the numerical proof of these conditions, or analytical proof in interesting asymptotic limits (in the spirit of [27]), as important open problems for the theory. The determination of asymptotic behavior in the small-wavelength limit, analogously as in [15] for the reaction-diffusion case, is another important open problem. Likewise, extensions to the case that not only the phase but the wave number $\kappa = \Psi_x$ has different values at plus and minus infinity, corresponding to Riemann data for the Whitham equation (1.7), is an interesting direction for future investigation; see [15, 8] in the reaction-diffusion case.

Plan of the paper: The plan of the rest of the paper is as follows. In Section 2 we set up the framework of the proofs (introduction of the phase, integral transform, etc.), then in Section 3 we prove Theorem 1.10. Finally, in Section 4, we give the proof of Theorem 1.12. In Appendices A–D we provide, respectively, algebraic relations obtained by differentiation of the traveling-wave ODE, derivation of the Whitham system (1.7), simplifications of this system afforded by the theory of

[20]Recall that $\mathcal{M}$ does not appear in the Whitham equation for the reaction-diffusion case.

parabolic conservation laws for data consisting of localized perturbations of a constant state, and some hints regarding generalizations to other situations of mathematical or physical interest.

Outline of the proof(s): We end this introduction by providing the reader with a guide through the analysis. At a first (nontechnical) glance, our strategy is essentially similar to the one of [25, 26] and we again encourage the reader to consult those references. It consists first in introducing $v(x,t)=\tilde{u}(x-\psi(x,t),t)-\bar{U}(x)$ and ψ. At the linearized level, the system becomes $(\partial_t-L)(v+\psi\bar{U}')=0$; hence the integral formulation

$$v(t)+\psi(t)\bar{U}' \;=\; e^{t\,L}(v(0)+\psi(0)\bar{U}')+\int_0^t e^{(t-s)\,L}\mathcal{N}(s)\,ds,$$

where $\mathcal{N}$ denotes nonlinear terms. We decompose this equation through a semi-group splitting stemming from the introduction of $(n+1)$ critical evolutions $s_j^{\mathrm{p}}(t)$ accounting for the $(n+1)$ critical spectral curves passing through 0 at $\xi=0$. These we treat always as a critical block $s^{\mathrm{p}}(t)=\sum_{j=1}^{n+1}s_j^{\mathrm{p}}(t)$. To prove Theorem 1.10, we retain from this critical evolution only the contribution to phase modulation which results, with our notation below, in the splitting $e^{t\,L}=\bar{U}'e_{n+1}\cdot s^{\mathrm{p}}(t)+\tilde{S}(t)$ with $\tilde{S}(t)$ faster-decaying and preserving localization. With this in hand, in order to accomodate both the localization defects coming from the emergence of diffusion-waves in parameters and from the initial defects (h_0 is not mean-free), we choose as an equation for ψ an interpolation of

$$\psi(t) \;=\; \begin{cases} e_{n+1}\cdot s^{\mathrm{p}}(t)(v(0)+h_0\bar{U}')+\displaystyle\int_0^t e_{n+1}\cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)\,ds & \text{for } t\geq 1\\ h_0 \qquad \text{for } t\leq 1/2 \end{cases}$$

allowing us, after differentiation, to work directly with localized quantities (v,ψ_x,ψ_t). The proof of Theorem 1.10 is then achieved by first closing nonlinear estimates with non-sharp decay rates and then improving these decay rates in low-regularity norms. In this process, estimates on v essentially[21] follow from energy estimates on the original system, either through a nonlinear high-frequency damping inequality or through the fact that their counterparts at the resolvent level enable us to convert spectral assumptions into exponential decay for the non-critical part of the semi-group provided we restrict ourselves to Hilbert-space estimations. Modulation parameters being inherently slowly-evolving, or more properly speaking of low-frequency type, regularity is not an issue for ψ and its derivatives. Nevertheless estimates on ψ do require a much more involved and subtle space/frequency analysis. In particular, a great care is needed to ensure that contributions of $h_0\bar{U}'$ may indeed be bounded in terms of norms of $\partial_x h_0$ and that no localization is lost in the time-transition layer. The proof of Theorem 1.12 requires the use of the full critical block $s^{\mathrm{p}}(t)$ to get a refined description of $e^{t\,L}$ leading to $v=\partial_k\bar{U}\,\bar{k}\psi_x+\partial_M\bar{U}\cdot M+z$, with z faster-decaying and

$$\begin{pmatrix} M(t)\\ \bar{k}\psi(t)\end{pmatrix} \;=\; \bar{k}\partial_x s^{\mathrm{p}}(t)(v(0)+h_0\bar{U}')+\int_0^t \bar{k}\partial_x s^{\mathrm{p}}(t-s)\mathcal{N}(s)\,ds \qquad \text{for } t\geq 1.$$

To make the connection with (1.7), we remark that, by subtituting time derivatives by space derivatives thanks to the analytical validation of the role of linear group velocities, we may write nonlinear terms as a quadratic function of $(M,\bar{k}\psi_x)$ with periodic coefficients, plus a faster-decaying remainder. In particular, we only need then to make the connection at the level of evolution operators. The analytical connection between $s^{\mathrm{p}}(t)$ and the linearized evolution of (1.7), which again involves some intricate analysis, reveals in which way data for the original system are translated into data for the Whitham system. Finally, having checked that all relevant terms match by computing needed averages, we prove the nonlinear connection to (1.7) by closing nonlinear estimates in a standard way.

[21] Up to the contribution of the critical part of the evolution not described by phase modulation but estimated in a similar way.

2. Preliminaries

In this section, we discuss several technical preliminaries that we will find useful throughout our analysis. Specifically:

- we introduce the Bloch transform, the fundamental integral transform that we will use in deriving all of our linear estimates;
- we show how phase shift ψ is introduced in (1.1) and how this affects the equations;
- we prove a nonlinear damping energy estimate (here simply following from the parabolic nature of (1.1)), establishing that high derivatives of the solution decay in time at least as fast as low derivatives so that technical issues are mainly, as expected, in decay rates and localization (small Floquet or small Fourier numbers) and not in regularity;
- we give estimates useful to analyze the effect of a change of independent variables on our bounds.

These preliminary issues were already present implicitly or explicitly in [29, 30].

2.1. **Bloch decomposition.** To begin our analysis of the stability of a fixed 1-periodic stationary solution $\bar{U}$ of (1.1), recall from above that linearizing the flow of (1.1) about $\bar{U}$ leads to the consideration of the 1-periodic coefficient linear evolution equation (1.10). From Floquet theory, one may guess that it would be desirable in analyzing this equation to decompose solutions as superpositions of functions having a given Floquet exponent $\xi \in [-\pi, \pi]$, i.e., functions $e^{i\xi\cdot}h(\cdot)$ with $h \in L^2([0,1])_{\mathrm{per}}$, as described in Remark 1.3. This may be accomplished using the *Bloch transform.*

Given a function $g \in L^2(\mathbb{R})$, its *Bloch decomposition*, or inverse Bloch transform representation, is defined as

$$g(x) = \int_{-\pi}^{\pi} e^{i\xi x}\check{g}(\xi, x)d\xi, \tag{2.1}$$

where

$$\check{g}(\xi, x) := \sum_{k\in\mathbb{Z}} e^{2\pi i k x}\hat{g}(\xi + 2\pi k),$$

and $\hat{g}(z) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega z}g(\omega)d\omega$ is the Fourier transform of g. Note that for any $\xi \in [-\pi, \pi]$, $\check{g}(\xi, \cdot)$ is a 1-periodic function, hence, as desired, $e^{i\xi\cdot}\check{g}(\xi, \cdot)$ has Floquet exponent ξ.

Letting $\mathcal{B} : L^2(\mathbb{R}) \to L^2([-\pi, \pi]; L^2_{\mathrm{per}}([0,1]))$, $g \mapsto \check{g}$ denote the Bloch transform, we readily see that for our given linearized operator L, defined in (1.10), and $g \in L^2(\mathbb{R})$ we have $\mathcal{B}(Lg)(\xi, x) = L_\xi\left[\check{g}(\xi, \cdot)\right](x)$, hence the associated Bloch operators L_ξ may be viewed as operator-valued symbols under $\mathcal{B}$, acting on $L^2_{\mathrm{per}}([0,1])$. Similarly, from the identity $\mathcal{B}\left(e^{tL}g\right)(\xi, x) = \left(e^{tL_\xi}\check{g}(\xi, \cdot)\right)(x)$, a consequence of (1.11), we find the *Bloch solution formula* for the periodic-coefficient operator L:

$$(S(t)g)(x) := (e^{tL}g)(x) = \int_{-\pi}^{\pi} e^{i\xi x}(e^{tL_\xi}\check{g}(\xi, \cdot))(x)d\xi. \tag{2.2}$$

In particular, we see that the Bloch transform $\mathcal{B}$ diagonalizes the periodic coefficient operator L in the same way that the Fourier transform diagonalizes constant-coefficient operators.

Using the representation formula (2.2), bounds on the Bloch solution operator e^{tL_ξ} can be converted to bounds on the linearized solution operator e^{tL}. To facilitate these bounds, we notice by the standard Parseval identity that the rescaled Bloch transform $\sqrt{2\pi}\mathcal{B}$ is an isometry on $L^2(\mathbb{R})$, i.e.

$$\|g\|^2_{L^2(\mathbb{R})} = 2\pi\int_{-\pi}^{\pi}\int_0^1 |\mathcal{B}(g)(\xi, x)|^2 dx\ d\xi = 2\pi\|\check{g}\|^2_{L^2([-\pi,\pi];L^2([0,1]))}. \tag{2.3}$$

More generally, by interpolating (2.3) with the triangle inequality, corresponding to the case $q = 1$ and $p = \infty$ below, we obtain the generalized Hausdorff–Young inequality $\|g\|_{L^p(\mathbb{R})} \leq$

$\|\check{g}\|_{L^q([-\pi,\pi],L^p([0,1]))}$ for $q \le 2 \le p$ and $\frac{1}{p}+\frac{1}{q}=1$, which, by (2.1), yields for any 1-periodic functions $g(\xi,\cdot)$

$$(2.4) \qquad \Big\| \int_{-\pi}^{\pi} e^{i\xi\cdot} g(\xi,\cdot) d\xi \Big\|_{L^p(\mathbb{R})} \le \|g\|_{L^q([-\pi,\pi],L^p([0,1]))} \text{ for } q \le 2 \le p \text{ and } \frac{1}{p}+\frac{1}{q}=1.$$ [22]

It is from this convenient formulation that we will obtain our linear estimates.

Remark 2.1. To keep technicalities as low as possible, we have indeed compelled ourselves to prove all our linear estimates using (2.4) alone. The only price to pay is that we are thus confined to high-norm estimates, in $W^{k,p}$, $2 \le p \le \infty$, as a consequence of which, due to the details of the nonlinear iteration estimates, bound (1.16)(i) (thus (1.19)(i)) is not expected to be sharp except for $p=2$. With more work, but in the spirit of the present paper, we expect that one may actually prove a $(1+t)^{-1/2(1-1/p)-1/2+\eta}$ decay ($\eta > 0$ arbitrary). See Remarks 1.7 and 4.3 of [30] for further discussion in the somewhat simpler setting of the reaction-diffusion case.

Remark 2.2. Another important property of the Bloch transform that we will use repeatedly throughout our proofs is that it is well-behaved with respect to 2-scale analysis of a slow modulation *ansatz*. Indeed if g is 1-periodic and h is slow, in the sense that the Fourier transform of h is supported in $[-\pi,\pi]$, then $\mathcal{B}(gh)(\xi,x) = g(x)\widehat{h}(\xi)$. If no slowness assumption is made but still g is 1-periodic, then there still holds $\mathcal{B}(gh)(\xi,x) = g(x)\check{h}(\xi,x)$. Mark that the fact that in general high frequencies of h are involved in the low Floquet number part of gh will add substantial difficulties to the linear analysis below. This nice property that the Bloch transform separates scales may also be used to perform two separate change of frames for fast and slow variables through $\check{g}(\xi,x) \mapsto e^{i\xi c_1 t}\check{g}(\xi, x-c_2 t)$. This feature was used in a crucial way in the analysis of reaction-diffusion systems carried out in [57], where the proof relied strongly on self-similar techniques such as renormalization, requiring the Whitham equation to be essentially reduced to a viscous Burgers equation by going into the frame of its characteristic velocity (called linear group velocity) while keeping the original equation in the co-moving frame of the wave (the one of the phase velocity). This elegant strategy appears to completely break down in the system case considered here, for which many linear group velocities are involved.

2.2. **Nonlinear perturbation equations.** We now discuss how the introduction of a phase shift affects system (1.1). Following [29, 25], we introduce the perturbation variable

$$(2.5) \qquad v(x,t) = \tilde{u}(x-\psi(x,t),t) - \bar{U}(x)$$

where $\tilde{u}(x,t)$ satisfies (1.1) and $\psi(x,t)$ is a phase shift to be determined together with v. We recall the following representation convenient for nonlinear iteration and established in [29].

Lemma 2.3 ([29]). *The nonlinear residual v and phase shift ψ linked by* (2.5) *satisfy*

$$(2.6) \qquad (\partial_t - L)(v+\psi\bar{U}_x) = \mathcal{N}, \qquad \textit{with} \qquad \mathcal{N} := \partial_x\mathcal{Q} + \partial_x\mathcal{R} + \partial_t\mathcal{S},$$

where

$$(2.7) \qquad \mathcal{Q} := -\bar{k}\left(f(\bar{U}+v) - f(\bar{U}) - df(\bar{U})v\right),$$

$$(2.8) \qquad \mathcal{R} := -v\psi_t + \bar{k}^2 v_x \frac{\psi_x}{1-\psi_x} + \bar{k}^2\bar{U}_x \frac{\psi_x^2}{1-\psi_x},$$

and

$$(2.9) \qquad \mathcal{S} := v\psi_x.$$

[22]Here and elsewhere, we denote $\|g\|_{L^q([-\pi,\pi],L^p([0,1]))} := \left(\int_{-\pi}^{\pi} \|g(\xi,\cdot)\|_{L^p([0,1])}^q d\xi\right)^{1/q}$.

Remark 2.4. As noted in [29], the advantage of (2.5), $\tilde{u}(\cdot,t) = (\bar{U} + v(\cdot,t)) \circ \Psi(\cdot,t)$ (with $\Psi(\cdot,t) = (\mathrm{Id} - \psi(\cdot,t))^{-1}$) over the (probably more natural) choice $\tilde{u}(\cdot,t) = \bar{U} \circ \Psi(\cdot,t) + \tilde{v}(\cdot,t)$ suggested by (1.6), is that the phase shift enters the equation only through commutators between composition with Ψ and differentiation, hence only gradients of ψ appear in the source terms $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ on the right-hand side of (2.6). By contrast, the corresponding terms for choice $\tilde{v}$ would involve also terms of order $|\psi||\psi_x|$ including the nondecaying phase ψ itself, thus making the decay too slow for our nonlinear iteration to close. This observation is by now classical in the stability analysis of traveling patterns; see the related observations of [15] regarding stability of periodic reaction-diffusion waves, and [71, 72, 42, 23] and [65, 66, 67, 73][23] for similar, earlier, observations in the context of viscous shock stability and bifurcation.

However, notice also the more subtle aspect of the decomposition 2.6 that it groups within the term $(\partial_t - L)(\psi \bar{U}_x)$ appearing on the left-hand side the linear order source terms $\psi_t \bar{U}_x$ and $\psi_x \bar{U}_{xx}$ that are individually too large to handle in our later nonlinear iteration; the term $(\partial_t - L)(\psi \bar{U}_x)$ is then used to cancel instantaneous phase-modulations arising in the solution of the linearized equations, as described just below. This approach to detecting nonlinear cancellation originates from the study of stability of viscous shock solutions of systems of parabolic conservation laws (see, e.g., [71, Eq. (2.30)], [72], and especially [23, Eq. (5.23), Cor. 5.4, p. 453]) and is fundamentally different from those introduced in the reaction-diffusion setting in [15, 57] based on normal forms and successive reductions/renormalizations, which, as discussed in Remark 2.2(ii), appear unlikely to work in the present case. Indeed, we view this distinction as the key to the successful treatment in [29, 30] of nonlinear modulational stability in the presence of conserved quantities.

Isolation of the phase. To motivate the more technical analysis of Section 3, we describe in informal fashion the way that we determine the phase ψ, separating out principal nonlinear behavior. Using Duhamel's formula together with (2.6), we can write an integral equation for v as

$$v(\cdot,t) + \psi(\cdot,t)\bar{U}_x = e^{tL}(v(\cdot,0) + \psi(\cdot,0)\bar{U}_x) + \int_0^t e^{(t-s)L}\mathcal{N}(s)ds, \tag{2.10}$$

with initial data $\psi(\cdot,0) = h_0$, $v(\cdot,0) = \tilde{u}_0 \circ (\mathrm{Id} - h_0) - \bar{U}$. Noting, by the bounds of Section 3, that $e^{tL} = \bar{U}_x e_{n+1} \cdot s^{\mathrm{p}}(t) + \tilde{S}(t)$, where $s^{\mathrm{p}}(t)$ is some operator sending $\mathbb{R}^n$-valued functions into $\mathbb{R}^{n+1}$-valued functions[24] (e_{n+1} denoting the $(n+1)$th standard basis element) and $\tilde{S}(t)$ is a faster-decaying residual, that is, that the principal part of the linear solution operator is a linear phase-modulation consisting of $e_{n+1} \cdot s^{\mathrm{p}}(t)$ times the instantaneous shift $\bar{U}_x$, we remove this principal part by defining implicitly

$$\psi(t) \sim e_{n+1} \cdot s^{\mathrm{p}}(t)(v(\cdot,0) + \psi(\cdot,0)\bar{U}_x) + \int_0^t e_{n+1} \cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds, \tag{2.11}$$

where the $\sim$ here indicates equality for $t \geq 1$. This gives an expression

$$v(t) \sim \tilde{S}(t)(v(\cdot,0) + \psi(\cdot,0)\bar{U}_x) + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds \tag{2.12}$$

for v in which no $s^{\mathrm{p}}(t)$ terms appear, closing the system in (ψ, v).

This simple prescription follows the principle that we should choose the nonlinear phase so as to remove from the linear description of the residual v all contributions representing linearized, or "instantaneous" phase modulation, at least away from the initial layer $0 \leq t \leq 1$. On this latter time interval, where we are constrained by the restriction $\psi(\cdot,0) = h_0$, we instead interpolate between the right-hand side of (2.11) and the initial data $\psi(\cdot,0) = h_0$, as described in (3.11)–(3.13).

[23]Specifically, [65, Section 3] and [66, Section 2.2.1] (group invariance and uniqueness), [73, Section 3] (translation-invariant center–stable manifold), and [67, Theorem 2.2.0] (Nash–Moser uniqueness theorem).

[24]The first space is the one of U-values, the second one is the one of modulation parameters $(\mathcal{M}, \kappa)$.

2.3. **Nonlinear damping estimate.** To complement the linear bounds, established below, that form the core of the proof, we will use the following damping-type bound established by energy estimate in [29], useful in controlling higher derivatives by lower ones, enabling us to close a nonlinear iteration with decay rates of the lower derivatives.

Proposition 2.5 ([29])**.** *Assuming (H1)-(H3), there exist positive constants θ, C and ε such that if v and ψ solve* (2.6) *on* $[0,T]$ *for some* $T>0$ *and*

$$\sup_{t\in[0,T]}\|(v,\psi_x)(t)\|_{H^K(\mathbb{R})}+\sup_{t\in[0,T]}\|\psi_t(t)\|_{H^{K-1}(\mathbb{R})}\le\varepsilon$$

then, for all $0\le t\le T$,

$$\|v(t)\|^2_{H^K(\mathbb{R})}\le Ce^{-\theta t}\|v(0)\|^2_{H^K(\mathbb{R})}+C\int_0^t e^{-\theta(t-s)}\left(\|v(s)\|^2_{L^2(\mathbb{R})}+\|(\psi_t,\psi_x,\psi_{xx})(s)\|^2_{H^{K-1}(\mathbb{R})}\right)ds. \tag{2.13}$$

Proof. (from [29]) Rewriting (2.6) as

$$(1-\psi_x)v_t-\bar{k}^2v_{xx}=\bar{k}\bar{c}v_x-\psi_t(\bar{U}_x+v_x)-\bar{k}(f(\bar{U}+v)-f(\bar{U}))_x+\bar{k}\left(\frac{\bar{k}\psi_x}{1-\psi_x}(\bar{U}_x+v_x)\right)_x, \tag{2.14}$$

taking the $L^2(\mathbb{R})$ inner product against $\sum_{j=0}^K\dfrac{(-1)^j\partial_x^{2j}v}{1-\psi_x}$, integrating by parts, and rearranging, we obtain

$$\frac{d}{dt}\|v\|^2_{H^K(\mathbb{R})}(t)\le-\tilde{\theta}\|\partial_x^{K+1}v(t)\|^2_{L^2(\mathbb{R})}+\tilde{C}\left(\|v(t)\|^2_{H^K(\mathbb{R})}+\|(\psi_t,\psi_x,\psi_{xx})(t)\|^2_{H^{K-1}(\mathbb{R})}\right),$$

for some positive $\tilde{C}$ and $\tilde{\theta}$, so long as $\|(v,v_x,\psi_t,\psi_x,\psi_{xx})(t)\|_{H^{K-1}(\mathbb{R})}$ remains sufficiently small. Sobolev interpolation $\|g\|^2_{H^K(\mathbb{R})}\le a^{-1}\|\partial_x^{K+1}g\|^2_{L^2(\mathbb{R})}+a\|g\|^2_{L^2(\mathbb{R})}$ gives, then, for $a>0$ sufficiently large,

$$\frac{d}{dt}\|v\|^2_{H^K(\mathbb{R})}(t)\le-\theta\|v(t)\|^2_{H^K(\mathbb{R})}+C\left(\|v(t)\|^2_{L^2(\mathbb{R})}+\|(\psi_t,\psi_x)(t)\|^2_{H^K(\mathbb{R})}\right),$$

from which (2.13) follows by Gronwall's inequality. See [29] for further details. □

Remark 2.6. It should be pointed out that one technical feature of our nonlinear iteration is the loss of derivatives of the nonlinear perturbation variable v, i.e. our iteration argument shall control L^p norms of v in terms of H^s norms of v and gradients of ψ. This loss of derivatives is compensated by the above "nonlinear damping" estimate, which is due to the (here, total) parabolicity of the governing equations. Of course, here, other strategies would be available (such as maximal regularity). The advantage of the above energy estimate is that it generalizes to partially parabolic systems (through the introduction of Friedrichs symmetrizers with Kawashima compensators; see [33] for such an analysis in the context of the Saint-Venant system). In contrast, the fact that the Bloch transform involves only bounded Floquet numbers translates immediately into the fact that local parameters such as ψ (or M) should be "slow," in the sense of Remark 2.2, so that arbitrarily many derivatives are gained at linear level.

2.4. **Inverse modulation bounds.** An important technical detail [15, 25, 26], and a new issue beyond the viscous shock wave case mentioned in Remark 2.4, is the relation between quantities $\|\tilde{u}(\cdot-\psi(\cdot,t),t)-U^{\bar{M}+M(\cdot,t),\bar{k}/(1-\psi_x(\cdot,t))}(\cdot)\|_{L^p(\mathbb{R})}$ conveniently estimable by our analysis and the corresponding quantities $\|\tilde{u}(\cdot,t)-U^{\mathcal{M}(\cdot,t),\kappa(\cdot,t)}(\Psi(\cdot,t))\|_{L^p(\mathbb{R})}$ arising through formal Whitham approximation. With this in mind, we remark that the following (sharp) estimate shows that $\|F(\cdot)-G(\cdot+\phi(\cdot))\|_{L^p(\mathbb{R})}$ is essentially equivalent to $\|F(\cdot-\phi(\cdot))-G(\cdot)\|_{L^p(\mathbb{R})}$ plus $\|\phi\|_{L^\infty(\mathbb{R})}\|\phi_x\|_{L^p(\mathbb{R})}$.

Lemma 2.7. *Let ϕ be bounded with $\|\phi_x\|_{L^\infty(\mathbb{R})} < 1$. Then* $\mathrm{Id} - \phi$ *is invertible and*

$$
(2.15)\qquad
\begin{aligned}
\|F - G\circ(\mathrm{Id}-\phi)^{-1}\|_{L^p(\mathbb{R})} &\leq (1+\|\phi_x\|_{L^\infty(\mathbb{R})})^{\frac{1}{p}} \quad \|F\circ(\mathrm{Id}-\phi) - G\|_{L^p(\mathbb{R})}\\
\|F - G\circ(\mathrm{Id}+\phi)\|_{L^p(\mathbb{R})} &\leq (1+\|\phi_x\|_{L^\infty(\mathbb{R})})^{\frac{1}{p}} \quad \|F\circ(\mathrm{Id}-\phi) - G\|_{L^p(\mathbb{R})}\\
&\quad +\|G_x\|_{L^\infty(\mathbb{R})}(1+\|\phi_x\|_{L^\infty(\mathbb{R})})^{\frac{1}{p}}\|\phi\|_{L^\infty(\mathbb{R})}\|\phi_x\|_{L^p(\mathbb{R})}.
\end{aligned}
$$

Proof. By the implicit function theorem and boundedness of ϕ, the map $\mathrm{Id}-\phi$ is invertible. Let us write its inverse $\mathrm{Id}+\tilde{\phi}$. Since the Jacobian of $\mathrm{Id}+\tilde{\phi}$ is bounded below by $(1+\|\phi_x\|_{L^\infty(\mathbb{R})})^{-1}$, we remark that

$$
\|[F\circ(\mathrm{Id}-\phi) - G]\circ(\mathrm{Id}+\tilde{\phi})\|_{L^p(\mathbb{R})} \;\leq\; (1+\|\phi_x\|_{L^\infty(\mathbb{R})})^{\frac{1}{p}}\|F\circ(\mathrm{Id}-\phi)-G\|_{L^p(\mathbb{R})}
$$

and the first part of (2.15) follows. We then split $F - G\circ(\mathrm{Id}+\phi)$ as

$$
F - G\circ(\mathrm{Id}+\phi) \;=\; [F\circ(\mathrm{Id}-\phi) - G]\circ(\mathrm{Id}+\tilde{\phi}) \;+\; G\circ(\mathrm{Id}+\tilde{\phi}) - G\circ(\mathrm{Id}+\phi).
$$

Now the intermediate value theorem yields

$$
\|G\circ(\mathrm{Id}+\tilde{\phi}) - G\circ(\mathrm{Id}+\phi)\|_{L^p(\mathbb{R})} \;\leq\; \|G_x\|_{L^\infty(\mathbb{R})}\|\tilde{\phi}-\phi\|_{L^p(\mathbb{R})}.
$$

But, from equality $\tilde{\phi} = \phi\circ(\mathrm{Id}+\tilde{\phi})$ we infer

$$
\tilde{\phi}(x) - \phi(x) \;=\; \tilde{\phi}(x)\int_0^1 \phi_x(x+t\tilde{\phi}(x))\,dt
$$

from which Hölder's inequality yields

$$
\|\tilde{\phi}-\phi\|^p_{L^p(\mathbb{R})} \;\leq\; \|\tilde{\phi}\|^p_{L^\infty(\mathbb{R})}\int_0^1 \|\phi_x\circ(\mathrm{Id}+t\tilde{\phi})\|^p_{L^p(\mathbb{R})}\,dt.
$$

This concludes the proof since $\|\tilde{\phi}\|_{L^\infty(\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R})}$ and, for $t\in[0,1]$, $\mathrm{Id}+t\tilde{\phi}$ is invertible with a Jacobian bounded below by $(1+\|\phi_x\|_{L^\infty(\mathbb{R})})^{-1}$. See [26, Remark 1.4] for related comments. □

Remark 2.8. Note that the quantity $\tilde{u}(\cdot - \psi(\cdot,t),t)$ estimated in our analysis does not necessarily control $\tilde{u}(\cdot,t)$ unless $\bar{k}\psi_x$ is small in L^∞ (local invertibility of $\mathrm{Id}-\psi$) and bounded in L^1 (global invertibility). *Thus, our approach* (and likewise that of [25, 26, 15, 57]) *is inherently a small-variation analysis in wave number,* whether or not the Whitham system admits large-variation solutions.[25]

However, if the Whitham system has an associated convex entropy [39, 64], then, for $\kappa-\bar{k}\sim\bar{k}\psi_x$ initially small in L^∞ and bounded in L^1 (hence small in L^2), by the results of [22], it has a solution that remains small in L^∞, and decays as $(1+t)^{-\frac{1}{2}(1-1/p)}$ in all L^p, whence, combining the stability and behavior arguments of Sections 3–4 and closing an iteration for z (the refined perturbation variable defined in (4.7)) instead of v, it might be possible to relax the assumption $\|\psi_x(0)\|_{L^1\cap L^\infty} \ll 1$ to $\|\psi_x(0)\|_{L^1} = \mathcal{O}(1)$ and $\|\psi_x(0)\|_{L^\infty} \ll 1$, allowing large variations in phase. In the absence of a convex entropy, one might instead assume closeness of $\psi_x(0)$ in $L^1\cap H^s$ to a special "diffusion wave" solution (in the scalar case, a distorted Gaussian obtained by Hopf–Cole transformation; in the system case, a more complicated coupled superposition of such waves [41]) that has evolved sufficiently long to be of small gradient, again relaxing slightly the restriction that ψ be of small initial variation. These would be interesting directions for future investigation.

[25] In particular, for the three example systems considered in the introduction, the Whitham system does have large-amplitude solutions for data merely bounded in $L^1\cap L^2$ since it has an associated convex entropy; see [22].

3. Stability

In this section, we prove Theorem 1.10. In particular, to separate further (2.10), we first need a precise spectral analysis (proving Proposition 1.7 along the way) that will allow for a separation of e^{tL} into a part aligned with $\bar{U}_x$ plus a faster-decaying term. Throughout this analysis, we shall often refer to algebraic relations obtained from the profile equation (1.3) and stored in Appendix A.

3.1. **Spectral analysis and nonlinear decomposition.**

Proof of Proposition 1.7. By (A.3), one may choose $(\partial_{M_1}\bar{U}, \dots, \partial_{M_n}\bar{U}, \bar{U}')$ as a right basis for Σ_0, and $(e_1, \dots, e_n, \bar{u}^{adj})$ as the dual left basis, where e_j denotes the constant function equal to the jth standard Euclidean basis element, and $\bar{u}^{adj}$ denotes a generalized zero eigenfunction of the adjoint operator $L_0^* = \bar{k}^2\partial_x^2 + \bar{k}(df(\bar{U}) - \bar{c})\partial_x$ such that $\langle \bar{u}^{adj}, \partial_M \bar{U}\rangle = 0$ and $\langle \bar{u}^{adj}, \bar{U}'\rangle = 1$. By standard spectral perturbation theory [35] we may build spectral projectors and appropriately extend (locally near $\xi = 0$) these dual bases in an analytic way into dual right and left bases $\{q_j(\xi)\}_{j=1}^{n+1}$ and $\{\tilde{q}_j(\xi)\}_{j=1}^{n+1}$ associated to the spectrum of L_ξ in some fixed neighborhood of the origin. This spectrum is then precisely the one of the matrix

$$\Lambda_\xi \;=\; \left(\langle \tilde{q}_j(\xi), L_\xi q_l(\xi)\rangle\right)_{j,l}, \tag{3.1}$$

which we expand as $\xi \to 0$ as

$$\Lambda_\xi \;=\; \Lambda_0 + (i\bar{k}\xi)\Lambda^{(1)} + (i\bar{k}\xi)^2\Lambda^{(2)} + \mathcal{O}(\xi^3).$$

We expand also $L_\xi = L_0 + (i\bar{k}\xi)L^{(1)} + (i\bar{k}\xi)^2 L^{(2)}$; see (A.2) for definitions of the $L^{(j)}$.

Note that, replacing, if necessary, simultaneously $q_{n+1}(\xi)$ with

$$q_{n+1}(\xi) - \xi\sum_{j=1}^{n}\langle \tilde{q}_j(0), \partial_\xi q_{n+1}(0)\rangle\; q_j(\xi)$$

and, for $j \neq n+1$, $\tilde{q}_j(\xi)$ with $\tilde{q}_j(\xi) + \xi\langle \tilde{q}_j(0), \partial_\xi q_{n+1}(0)\rangle\; \tilde{q}_{n+1}(\xi)$, we may assume without loss of generality that, for $j \neq n+1$,

$$\langle \tilde{q}_j(0), \partial_\xi q_{n+1}(0)\rangle = 0, \qquad \langle \partial_\xi \tilde{q}_j(0), q_{n+1}(0)\rangle = 0,$$

the second inequality stemming from the first by expanding to first order in ξ the duality relation $\langle \tilde{q}_j(\xi), q_{n+1}(\xi)\rangle = 0$. Now note also that since $L_0 q_{n+1}(0) = 0$, expanding to first order in ξ the fact that $L_\xi q_{n+1}(\xi)$ lies in $\Sigma_\xi := \mathrm{Span}\{q_j(\xi)\}_{j=1}^{n+1}$ the critical space of L_ξ, we find that $L_0\partial_\xi q_{n+1}(0) + (i\bar{k})L^{(1)}q_{n+1}(0)$ lies in Σ_0, the generalized kernel of L_0. Using now (A.3)(iii) yields that $L_0(\partial_\xi q_{n+1}(0) - (i\bar{k})\partial_k U_{|(\bar{M},\bar{k})})$ lies also in the generalized kernel of L_0, thus so does $\partial_\xi q_{n+1}(0) - (i\bar{k})\partial_k U_{|(\bar{M},\bar{k})}$. Orthogonality relations from above and (A.3)(iii) lead then to $\partial_\xi q_{n+1}(0) - (i\bar{k})\partial_k U_{|(\bar{M},\bar{k})} \in \mathbb{C}\bar{U}'$.

The above preparation yields the following representations for Λ_0 and the first- and second-order correctors of the matrix Λ_ξ:

$$\Lambda_0 = \begin{pmatrix} 0_{n\times n} & 0_{n\times 1} \\ \langle \tilde q_{n+1}(0), L_0 q_l(0)\rangle & 0 \end{pmatrix} = \begin{pmatrix} 0_{n\times n} & 0_{n\times 1} \\ \partial_M \omega|_{(\bar M,\bar k)} & 0 \end{pmatrix},$$

$$\begin{aligned}\Lambda^{(1)} &= \begin{pmatrix} \frac{1}{ik}\langle \partial_\xi \tilde q_j(0), L_0 q_l(0)\rangle + \langle \tilde q_j(0), L^{(1)} q_l(0)\rangle & 0_{n\times 1} \\ * & \frac{1}{ik}\langle \tilde q_{n+1}(0), L_0\partial_\xi q_{n+1}(0)\rangle + \langle \tilde q_{n+1}(0), L^{(1)} q_{n+1}(0)\rangle \end{pmatrix} \\ &= \begin{pmatrix} -\Big((\partial_M F)|_{(\bar M,\bar k)} - \bar c \mathrm{Id}\Big) & 0_{n\times 1} \\ * & -\bar k (\partial_k c)|_{(\bar M,\bar k)} \end{pmatrix},\end{aligned}$$

$$\Lambda^{(2)} = \begin{pmatrix} * & \frac{1}{(ik)^2}\langle \partial_\xi \tilde q_j(0), L_0 \partial_\xi q_{n+1}(0)\rangle + \frac{1}{ik}\langle \partial_\xi \tilde q_j(0), L^{(1)} q_{n+1}(0)\rangle \\ & +\frac{1}{ik}\langle \tilde q_j(0), L^{(1)}\partial_\xi q_{n+1}(0)\rangle + \langle \tilde q_j(0), L^{(2)} q_{n+1}(0)\rangle \\ * & * \end{pmatrix} = \begin{pmatrix} * & -(\partial_k F)|_{(\bar M,\bar k)} \\ * & * \end{pmatrix}.$$

Therefore, there is no loss of regularity in ξ under the scaled similarity transformation

$$\tilde\Lambda_\xi = \frac{1}{i\bar k\xi}\begin{pmatrix} \mathrm{Id}_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & i\bar k \xi \end{pmatrix} \Lambda_\xi \begin{pmatrix} \mathrm{Id}_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & (i\bar k\xi)^{-1} \end{pmatrix} \tag{3.2}$$

and

$$\tilde\Lambda_0 = \begin{pmatrix} -\Big((\partial_M F)|_{(\bar M,\bar k)} - \bar c\mathrm{Id}\Big) & -(\partial_k F)|_{(\bar M,\bar k)} \\ -\bar k(\partial_M c)|_{(\bar M,\bar k)} & -\bar k(\partial_k c)|_{(\bar M,\bar k)} \end{pmatrix}.$$

This gives the result in a straightforward way. For omitted details, we point to similar computations in [48, 49, 32]. Recall also that this result is proved at spectral level (i.e., not including matrix expansion or eigenvector information) by an Evans function approach in [62, 52]. □

The former proof provides more than stated in Proposition 1.7. We collect supplementary information in the following lemma.

Proposition 3.1 ([49])**.** *Assuming (H1)–(H3) and (D1)–(D3), there exist $\varepsilon_0 > 0$, $\xi_0 \in (0,\pi)$, $n+1$ analytic curves, $j = 1,\dots,n+1$, $\lambda_j : [-\xi_0,\xi_0] \to B(0,\varepsilon_0)$ such that for $\xi \in [-\xi_0,\xi_0]$*

$$\sigma(L_\xi)\cap B(0,\varepsilon_0) = \{\ \lambda_j(\xi) \mid j \in \{1,\dots,n+1\}\ \}$$

and for $\theta > 0$ as in (D2)

$$\lambda_j(\xi) = -i\bar k\xi a_j + (i\bar k\xi)^2 b_j + \mathcal{O}(|\xi|^3), \qquad a_j, b_j \quad \textit{real}, \quad \bar k^2 b_j \geq \theta, \tag{3.3}$$

and associated left and right eigenfunctions $\phi_j(\xi)$ and $\tilde\phi_j(\xi)$ satisfying pairing relations

$$\langle \tilde\phi_j(\xi), \phi_k(\xi)\rangle = i\bar k\xi\delta_k^j, \qquad 1\leq j,k\leq n+1, \tag{3.4}$$

obtained as

$$\begin{aligned} \phi_j(\xi) &= (i\bar k\xi)\sum_{l=1}^n \beta_l^{(j)}(\xi) q_l(\xi) + \beta_{n+1}^{(j)}(\xi)\quad q_{n+1}(\xi) \\ \tilde\phi_j(\xi) &= \sum_{l=1}^n \tilde\beta_l^{(j)}(\xi)\tilde q_l(\xi) + (i\bar k\xi)\tilde\beta_{n+1}^{(j)}(\xi)\quad \tilde q_{n+1}(\xi) \end{aligned} \tag{3.5}$$

where

- $(q_1(\xi),\dots,q_{n+1}(\xi))$ *and* $(\tilde q_1(\xi),\dots,\tilde q_{n+1}(\xi))$ *are dual bases of spaces associated to the spectrum of respectively* L_ξ *and its adjoint* L_ξ^* *in* $B(0,\varepsilon_0)$, *analytic in* ξ, *bifurcating from* $(\partial_{M_1}\bar U,\dots,\partial_{M_n}\bar U,\bar U')$ *and* $(e_1,\dots,e_n,\bar u^{adj})$ *at* $\xi=0$, *with* e_j *the constant function equal to the* j*th standard Euclidean basis element and* $\bar u^{adj}$ *a generalized zero eigenfunction of* L_0^* *such that* $\langle \bar u^{adj},\partial_M\bar U\rangle=0$ *and* $\langle \bar u^{adj},\bar U'\rangle=1$, *and chosen such that*

$$\langle \tilde q_j(0),\partial_\xi q_{n+1}(0)\rangle=0,\qquad \langle \partial_\xi\tilde q_j(0),q_{n+1}(0)\rangle=0,\qquad 1\le j\le n; \tag{3.6}$$

- $(\beta^{(1)}(\xi),\dots,\beta^{(n+1)}(\xi))$ *and* $(\tilde\beta^{(1)}(\xi),\dots,\tilde\beta^{(n+1)}(\xi))$ *are dual right and left eigenbases, analytic in* ξ, *of the matrix* $\tilde\Lambda_\xi$, *defined in* (3.1)-(3.2), *associated to eigenvalues* $\lambda_j(\xi)/(i\bar k\xi)$ *and in particular for* $\xi=0$ *they form dual right and left eigenbases associated to* a_j*s of* $d(F,-\omega)_{|(\bar M,\bar k)}-\bar c\mathrm{Id}$.

Proof. This is a direct consequence of the proof of Proposition 1.7 except for the conditions on $\lambda_j(\xi)$, a_j, b_j, which follow from (D2) and complex conjugate symmetry together with (H3). □

Remark 3.2. Scaling transform (3.2) is directly related to the fact that the equation for the time evolution of local wavenumber is obtained by differentiating once in space the equation for the local phase. Though unnecessary to prove Proposition 1.7 in the uncoupled case, since the proposition follows then already from an examination of $\Lambda^{(1)}$, the manipulation is correct regardless of the linear coupling assumption. Yet in the uncoupled case, we may also assume $\tilde\beta_j^{(n+1)}(0)=0$ and $\beta_{n+1}^j(0)=0$ for $j\neq n+1$. Then by replacing $\tilde\phi_{n+1}(\xi)$ with $(i\bar k\xi)^{-1}\tilde\phi_{n+1}(\xi)$ and, for $j\neq n+1$, $\phi_j(\xi)$ with $(i\bar k\xi)^{-1}\phi_j(\xi)$, we obtain dual critical bases, analytic in ξ. For localized data this difference translates at the linear level into different decay rates. Note also that, even in the linearly uncoupled case, manipulations similar to scaling must be performed to get information about eigenf unctions.

Next, following [30, 25], in view of (3.5)(i) with $q_{n+1}(0)=\bar U'$, we decompose the solution operator $S(t)=e^{tL}$ as

$$S(t)=S^{\mathrm p}(t)+\tilde S(t),\qquad S^{\mathrm p}(t)=\bar U'\,e_{n+1}\cdot s^{\mathrm p}(t),\qquad s^{\mathrm p}(t)=\sum_{j=1}^{n+1}s_j^{\mathrm p}(t) \tag{3.7}$$

with

$$(s_j^{\mathrm p}(t)g)(x)=\int_{-\pi}^{\pi}e^{i\xi x}\alpha(\xi)e^{\lambda_j(\xi)t}\frac{1}{i\bar k\xi}\beta^{(j)}(\xi)\langle\tilde\phi_j(\xi,\cdot),\check g(\xi,\cdot)\rangle_{L^2([0,1])}d\xi, \tag{3.8}$$

and

(3.9)

$$\begin{aligned}(\tilde S(t)g)(x):=&\int_{-\pi}^{\pi}e^{i\xi x}(1-\alpha(\xi))(e^{tL_\xi}\check g(\xi))(x)d\xi+\int_{-\pi}^{\pi}e^{i\xi x}\alpha(\xi)(e^{tL_\xi}\tilde\Pi(\xi)\check g(\xi))(x)d\xi\\&+\int_{-\pi}^{\pi}e^{i\xi x}\alpha(\xi)\sum_{j=1}^{n+1}e^{\lambda_j(\xi)t}\frac{(\phi_j(\xi,x)-e_{n+1}\cdot\beta^{(j)}(\xi)\,q_{n+1}(0,x))}{i\bar k\xi}\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}d\xi,\end{aligned}$$

where α is a smooth cutoff function such that $0\le\alpha\le1$, $\alpha(\xi)=1$ for $|\xi|\le\xi_0/2$ and $\alpha(\xi)=0$ for $|\xi|\ge\xi_0$, and

$$\Pi^{\mathrm p}(\xi):=\sum_{j=1}^{n+1}q_j(\xi)\langle\tilde q_j(\xi),\cdot\rangle_{L^2([0,1])}=\frac{1}{i\bar k\xi}\sum_{j=1}^{n+1}\phi_j(\xi)\langle\tilde\phi_j(\xi),\cdot\rangle_{L^2([0,1])},\quad \tilde\Pi(\xi):=\mathrm{Id}-\Pi^{\mathrm p}(\xi) \tag{3.10}$$

denote respectively the eigenprojection, defined for $|\xi|\le\xi_0$, onto the critical space

$$\Sigma_\xi=\mathrm{Span}\{\phi_j(\xi)\}_{j=1}^{n+1}$$

bifurcating from Σ_0 at $\xi = 0$, and its complementary projection.

To establish nonlinear stability, there are of course other natural splitting choices available. In particular, the fact that we have kept in (3.8) the full $\beta^{(j)}(\xi)$ instead of $\beta^{(j)}(0)$ alone will play a role only in the asymptotic behavior study.

Next, we transpose this linear decomposition to the nonlinear level. Recalling Lemma 2.3 and integral equation (2.10), we start with the equation

$$(\partial_t - L)(v + \psi \bar{U}') = \mathcal{N}, \qquad v|_{t=0} = d_0,\ \psi|_{t=0} = h_0,$$

for the nonlinear residual $v(x,t)$ and the phase shift $\psi(x,t)$ both introduced in (2.5), where $d_0 := \tilde{u}_0(\cdot - h_0(\cdot)) - \bar{U} \in L^1(\mathbb{R}) \cap H^K(\mathbb{R})$, $\partial_x h_0 \in L^1(\mathbb{R}) \cap H^K(\mathbb{R})$, and notice, as in (2.10), that, after denoting solution operator $S(t) := e^{tL}$, an application of Duhamel's formula leads to

$$v(t) + \psi(t)\bar{U}' \ = \ S(t)(d_0 + h_0\bar{U}') + \int_0^t S(t-s)\mathcal{N}(s)ds. \tag{3.11}$$

In order to simultaneously accommodate the initial datum constraint $\psi(0) = h_0$ and absorb as much as possible $e_{n+1} \cdot s^{\mathrm{p}}(t)$ contributions into the equation for ψ, as described in the discussion surrounding (2.11)–(2.12), we split (3.11) as

$$\begin{aligned} \psi(t) &= e_{n+1} \cdot s^{\mathrm{p}}(t)(h_0\bar{U}' + d_0) + \int_0^t e_{n+1} \cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds \\ &\quad - (1-\chi(t))\left(e_{n+1} \cdot s^{\mathrm{p}}(t)(d_0 + h_0\bar{U}') - h_0 + \int_0^t e_{n+1} \cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\right), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} v(t) &= \tilde{S}(t)(d_0 + h_0\bar{U}') + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds \\ &\quad + (1-\chi(t))\left(S^{\mathrm{p}}(t)d_0 + (S^{\mathrm{p}}(t) - \mathrm{Id})(h_0\bar{U}') + \int_0^t S^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\right), \end{aligned} \tag{3.13}$$

where $\chi(t)$ is a smooth cutoff that is zero for $t \le 1/2$ and one for $t \ge 1$. We may extract from (3.12)-(3.13) a closed system in (v, ψ_x, ψ_t) (and some of their derivatives), and then recover ψ through the slaved equation (3.12).

We proceed stating now the linear estimates needed to bound the terms appearing in (3.12)-(3.13).

3.2. Basic linear estimates.

Proposition 3.3. *Under assumptions (H1)-(H3) and (D1)-(D3), for all $t \ge 0$, $2 \le p \le \infty$, and any $l, m \ge 0$, $r \ge 1$, $1 \le j \le n+1$,*

$$\left\| \partial_x^l \partial_t^m s_j^{\mathrm{p}}(t) g \right\|_{L^p(\mathbb{R})} \lesssim \min \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)+\frac{1}{2}-\frac{l+m}{2}} \|g\|_{L^1(\mathbb{R})} \\ (1+t)^{-\frac{1}{2}(1/2-1/p)+\frac{1}{2}-\frac{l+m}{2}} \|g\|_{L^2(\mathbb{R})} \end{cases} \tag{3.14}$$

when $l + m \ge 1$,

$$\| \, s_j^{\mathrm{p}}(t) g \, \|_{L^\infty(\mathbb{R})} \ \lesssim \ \|g\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})} \tag{3.15}$$

when ($l = m = 0$ and $p = \infty$), for some $\eta > 0$

$$\| \, e_{n+1} \cdot s^{\mathrm{p}}(t) g \, \|_{L^p(\mathbb{R})} \ \lesssim \ \min \begin{cases} (1+t) \quad \|g\|_{L^1(\mathbb{R})} \\ e^{-\eta t} \ \|g\|_{L^2(\mathbb{R})} \ + \ t^{\frac{1}{p}} \ \|g\|_{L^1(\mathbb{R})} \end{cases} \tag{3.16}$$

when ($l=m=0$ *and* $2\le p\le\infty$)*, and*

$$\left\|\partial_x^l\partial_t^m s_j^{\mathrm{p}}(t)\partial_x^r g\right\|_{L^p(\mathbb{R})}\lesssim\min\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac{l+m}{2}}\|g\|_{L^1(\mathbb{R})}\\(1+t)^{-\frac12(1/2-1/p)-\frac{l+m}{2}}\|g\|_{L^2(\mathbb{R})}\end{cases},\tag{3.17}$$

while, for some $\eta>0$, $0\le l+2m\le K+1$, *and* $2\le p\le\infty$,

$$\left\|\partial_x^l\partial_t^m\tilde S(t)g\right\|_{L^p(\mathbb{R})}\lesssim\min\begin{cases}e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})}+(1+t)^{-\frac12(1-1/p)}\|g\|_{L^1(\mathbb{R})}\\e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})}+(1+t)^{-\frac12(1/2-1/p)}\|g\|_{L^2(\mathbb{R})}\end{cases},\tag{3.18}$$

and, for $1\le r\le K+1$,

$$\left\|\partial_x^l\partial_t^m\tilde S(t)\partial_x^r g\right\|_{L^p(\mathbb{R})}\lesssim\min\begin{cases}e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})}+(1+t)^{-\frac12(1-1/p)-\frac12}\|g\|_{L^1(\mathbb{R})}\\e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})}+(1+t)^{-\frac12(1/2-1/p)-\frac12}\|g\|_{L^2(\mathbb{R})}\end{cases}.\tag{3.19}$$

Proof. (i) (Proof of (3.14)*).* First, notice that

$$(\partial_x^l\partial_t^m s_j^{\mathrm{p}}(t)g)(x)=\int_{-\pi}^{\pi}\alpha(\xi)e^{\lambda_j(\xi)t}e^{i\xi x}\frac{1}{i\bar k\xi}(i\xi)^l\lambda_j(\xi)^m\beta^{(j)}(\xi)\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}d\xi.\tag{3.20}$$

In the case $l+m\ge1$, estimates on s^{p} follows from, choosing either $s=1$ or $s=2$ fixed and introducing s' such that $1/s+1/s'=1$, the generalized Hausdorff-Young inequality (2.4) by

$$\begin{aligned}&\left\|\ x\mapsto\int_{-\pi}^{\pi}e^{i\xi x}\alpha(\xi)e^{\lambda_j(\xi)t}\frac{1}{i\bar k\xi}(i\xi)^l\lambda_j(\xi)^m\beta^{(j)}(\xi)\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}d\xi\ \right\|_{L^p(\mathbb{R})}\\&\lesssim\|\ (\xi,x)\mapsto\alpha(\xi)e^{\lambda(\xi)t}|\xi|^{l+m-1}|\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}|\ \|_{L^q([-\pi,\pi],L^p([0,1]))}\\&\lesssim\|\xi\mapsto|\xi|^{l+m-1}e^{-\eta\xi^2t}\|_{L^{r(s,p)}([-\pi,\pi])}\ \|\xi\mapsto\alpha(\xi)^{1/2}|\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}|\|_{L^{s'}([-\pi,\pi])}\\&\lesssim(1+t)^{-\frac12(1/s-1/p)-\frac{l+m-1}{2}}\|\xi\mapsto\alpha(\xi)^{1/2}|\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}|\|_{L^{s'}([-\pi,\pi])},\end{aligned}\tag{3.21}$$

where $1/p+1/q=1$ and $1/s'+1/r(s,p)=1/q$, so that $1/r(s,p)=1/s-1/p$. Here we have used (D2) to get for some $\eta>0$, $|e^{\lambda_j(\xi)t}\alpha^{1/2}(\xi)|\le e^{-\eta\xi^2t}$ and (1.12) to get $\lambda_j(\xi)=\mathcal{O}(\xi)$.

Now, for $s=2$, we note by the Cauchy-Schwarz inequality and Parseval identity (2.3) that

$$\begin{aligned}\left\|\xi\mapsto\alpha(\xi)^{1/2}|\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}\right\|_{L^2([-\pi,\pi])}&\le\sup_{|\xi|\le\xi_0}\|\tilde\phi_j(\xi)\|_{L^2([0,1])}\ \|\check g\|_{L^2([-\pi,\pi],L^2([0,1])}\\&\lesssim\sup_{|\xi|\le\xi_0}\|\tilde\phi_j(\xi)\|_{L^2([0,1])}\ \|g\|_{L^2(\mathbb{R})},\end{aligned}$$

where ξ_0 is given as in Proposition 3.1. For $s=1$ on the other hand, we begin by expanding

$$\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}=\sum_{j'\in\mathbb{Z}}\hat{\tilde\phi}_j(\xi,j')^*\hat g(\xi+2j'\pi),$$

where $\widehat{\tilde\phi}_j(\xi,j')$ denotes the j'th Fourier coefficient in the Fourier expansion of 2π-periodic function $\tilde\phi_j(\cdot)$, and $z^*=\bar z$ denotes complex conjugate. Applying the standard Hausdorff-Young inequality for the Fourier transform, i.e. $\|\hat g\|_{L^\infty(\mathbb{R})}\le\|g\|_{L^1(\mathbb{R})}$, together with the estimate

$$\alpha^{1/2}(\xi)\sum_{j'}|\widehat{\tilde\phi}_j(\xi,j')^*|\le\alpha^{1/2}(\xi)\sqrt{\sum_{j'}(1+|j'|^2)|\widehat{\tilde\phi}_j(\xi,j')|^2\sum_{j'}(1+|j'|^{-2})}\le C\alpha^{1/2}(\xi)\|\tilde\phi_j(\xi)\|_{H^1([0,1])},$$

which readily follows from the Cauchy–Schwarz' inequality, we obtain the bound

$$\|\xi\mapsto\alpha(\xi)^{1/2}|\langle\tilde\phi_j(\xi),\check g(\xi)\rangle_{L^2([0,1])}|\|_{L^\infty([-\pi,\pi])}\lesssim\sup_{|\xi|\le\xi_0}\|\tilde\phi_j(\xi)\|_{H^1([0,1])}\ \|g\|_{L^1(\mathbb{R})}.$$

Together with (3.21), this establishes (3.14).

(ii) (Proof of (3.15)*).* Thanks to (3.3), with the same kind of estimates as above one can bound in $L^\infty(\mathbb{R})$ the difference between $s_j^{\mathrm{p}}(t)(g)$ and

$$x \longmapsto \int_{-\pi}^{\pi} e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}e^{i\xi x}\frac{1}{i\bar{k}\xi}\beta^{(j)}(0)\langle\tilde{\phi}_j(0),\check{g}(\xi)\rangle_{L^2([0,1])}d\xi$$

by $C(1+t)^{-\frac{1}{2}}\|g\|_{L^1(\mathbb{R})}$. Since $\tilde{\phi}_j(0)$ is constant equal to $\tilde{\nu}_j := (\tilde{\beta}_1^{(j)}(0),\dots,\tilde{\beta}_n^{(j)}(0))$,

$$\langle\tilde{\phi}_j(0),\check{g}(\xi)\rangle_{L^2([0,1])} = \tilde{\nu}_j\cdot\widehat{g}(\xi)$$

and the last term is recognized to be the convolution of $\tilde{\nu}_j\cdot g$ with

$$\begin{aligned} x \;\mapsto\; \text{p.v.}\int_{-\pi}^{\pi} e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}e^{i\xi x}\frac{1}{i\bar{k}\xi}\beta^{(j)}(0)d\xi &= \text{p.v.}\int_{\mathbb{R}} e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}e^{i\xi x}\frac{1}{i\bar{k}\xi}\beta^{(j)}(0)d\xi \\ &\quad - \int_{\mathbb{R}\setminus[-\pi,\pi]} e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}e^{i\xi x}\frac{1}{i\bar{k}\xi}\beta^{(j)}(0)d\xi. \end{aligned}$$

By using the Cauchy-Schwarz inequality, we may bound the last integral in $L^2(\mathbb{R})$ with $Ce^{-\eta t}$, for some $\eta>0$. The remaining principal value is explicitly computed as a Gaussian error function

$$\frac{2\pi}{\bar{k}}\beta^{(j)}(0)\ \mathrm{errfn}\left(\frac{x+a_j\bar{k}t}{\sqrt{4\bar{k}^2b_jt}}\right)$$

and thus is bounded in $L^\infty(\mathbb{R})$. This achieves the proof of (3.15).

(iii) (Proof of (3.16)*).* We first remark that one can bound in $L^p(\mathbb{R})$ the difference between $e_{n+1}\cdot s^{\mathrm{p}}(t)(g)$ and

$$x \longmapsto \int_{-\pi}^{\pi} \quad e^{i\xi x}\sum_{j=1}^{n+1} e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}\frac{1}{i\bar{k}\xi}e_{n+1}\cdot\beta^{(j)}(0)\ \tilde{\nu}_j\cdot\widehat{g}(\xi)d\xi$$

by $C(1+t)^{-\frac{1}{2}(1-1/p)}\|g\|_{L^1(\mathbb{R})}$. Since $\{\beta_j(0)\}_j$ and $\{\tilde{\beta}_j(0)\}_j$ are dual bases, it follows, with $\tilde{\nu}_j$ still denoting the constant value of $\tilde{\phi}_j(0)$, that

$$\sum_{j=1}^{n+1}\beta^{(j)}(0)\ \tilde{\nu}_j^T \;=\; \sum_{j=1}^{n+1}\beta^{(j)}(0)\tilde{\beta}^{(j)}(0)^T\ \begin{pmatrix}\mathrm{Id}_{d\times d}\\ 0\cdots 0\end{pmatrix} \;=\; \begin{pmatrix}\mathrm{Id}_{d\times d}\\ 0\cdots 0\end{pmatrix},$$

and therefore

$$e_{n+1}\cdot\sum_{j=1}^{n+1}\beta^{(j)}(0)\ \tilde{\nu}_j^T \;=\; \begin{pmatrix}0 & \cdots & 0\end{pmatrix}.$$

Thus, the quantity to bound may be written as the convolution of g with the kernel

$$x \longmapsto \int_{-\pi}^{\pi} \quad e^{i\xi x}\sum_{j=1}^{n}\frac{e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}-e^{(-i\bar{k}a_{j+1}\xi-\bar{k}^2b_{j+1}\xi^2)t}}{i\bar{k}\xi}\ \left(\sum_{j'=1}^{j}e_{n+1}\cdot\beta^{(j')}(0)\ \tilde{\nu}_{j'}^T\right)\ d\xi. \tag{3.22}$$

Observing that

$$\left|\frac{e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}-e^{(-i\bar{k}a_{j+1}\xi-\bar{k}^2b_{j+1}\xi^2)t}}{i\bar{k}\xi}\right| \;\lesssim\; t$$

provides (3.16)(i).

To get (3.16)(ii), we first write the kernel (3.22) above as

$$x \longmapsto \int_{-\pi}^{\pi}\int_{-1}^{1} \quad \sum_{j=1}^{n} e^{i\xi\left(x+\bar{k}\left(\frac{a_j+a_{j+1}}{2}+\tau\frac{a_j-a_{j+1}}{2}\right)t\right)} \; t\left[\frac{a_j-a_{j+1}}{2}+i\bar{k}\xi(b_j-b_{j+1})\tau\right]$$
$$\times\, e^{-\left(\frac{b_j+b_{j+1}}{2}+\tau^2\frac{b_j-b_{j+1}}{2}\right)\bar{k}^2\xi^2 t} \left(\sum_{j'=1}^{j} e_{n+1}\cdot\beta^{(j')}(0)\;\tilde{\nu}_{j'}^{T}\right)\; d\tau d\xi.$$

Integration by parts in the ξ variable shows that for any fixed $\varepsilon>0$ the contribution to the $L^p(\mathbb{R})$ norm of the kernel of points x lying outside

$$I_t^{\varepsilon} \;:=\; \left[\; t\,\left(\min_{1\leq j\leq n+1} a_j-\varepsilon\right)\;,\; t\,\left(\max_{1\leq j\leq n+1} a_j+\varepsilon\right)\;\right]$$

is bounded by

$$C_\varepsilon\; t\;\left(\int_{\varepsilon t}^{\infty}\frac{dy}{y^p}\right)^{1/p} \;\leq\; C_\varepsilon t^{1/p}.$$

Following the end of the proof of (3.15), we may write the remaining part of the convolution as the convolution of g with a term decaying exponentially in time in $L^2(\mathbb{R})$ plus its convolution with a bounded function supported on I_t^{ε}. This yields (3.16)(ii).

(iv) (Proof of (3.17)*).* The proof goes similarly as the one of (3.14) thanks to the fact that since $\tilde{\phi}_j(0)$ is constant, for $r\geq 1$, the equality

$$\langle\tilde{\phi}_j(\xi),(\partial_x^r g)\check{}\,(\xi)\rangle_{L^2([0,1])} = \sum_{r'=0}^{r}\binom{r}{r'}(i\xi)^{r'}\langle\tilde{\phi}_j(\xi),\partial_x^{r-r'}\check{g}(\xi)\rangle_{L^2([0,1])}$$
$$=\xi(-1)^r\left\langle\partial_x^r\left(\frac{\tilde{\phi}_j(\xi)-\tilde{\phi}_j(0)}{\xi}\right),\check{g}(\xi)\right\rangle + i\xi\sum_{r'=1}^{r}\binom{r}{r'}(i\xi)^{r'-1}(-1)^{r-r'}\langle\partial_x^{r-r'}\tilde{\phi}_j(\xi),\check{g}(\xi)\rangle$$

holds and provides the factor ξ needed to compensate for $(i\bar{k}\xi)^{-1}$.

(v) (Proof of (3.18) *and* (3.19)*).* The last part of (3.9) is bounded as s_j^{p}, with an extra ξ factor compensating for $(i\bar{k}\xi)^{-1}$ thus enhancing decay and allowing for $l=m=0$ and $2\leq p\leq\infty$. We focus on the two remaining terms of (3.9).

To treat these, it is convenient to introduce on $H^l([0,1])$ a family of equivalent norms parametrized by $\xi\in[-\pi,\pi]$,

$$\|g\|^2_{\dot{H}^l_\xi([0,1])} := \sum_{j=0}^{l}\|(\partial_x+i\xi)^j g\|^2_{L^2([0,1])}$$

so that Parseval's identity implies

$$\|g\|^2_{H^l(\mathbb{R})} \;=\; (2\pi)\;\|\xi\mapsto\;\|\check{g}(\xi)\|_{\dot{H}^l_\xi([0,1]))}\;\|^2_{L^2([-\pi,\pi])}.$$

Now, thanks to standard resolvent bounds [21], assumptions (D1)-(D3) and the fact that $H^{l+2m+1}([0,1])$ and $L^2([0,1])$ spectra coincide [19], we may use Prüss' Theorem [56] and obtain that for some $\eta>0$

$$|e^{tL_\xi}(1-\alpha(\xi))|_{\dot{H}^{l+2m+1}_\xi([0,1])\to\dot{H}^{l+2m+1}_\xi([0,1])},\quad |\alpha(\xi)e^{tL_\xi}\;\tilde{\Pi}(\xi)|_{\dot{H}^{l+2m+1}_\xi([0,1])\to\dot{H}^{l+2m+1}_\xi([0,1])}\lesssim e^{-\eta t}.$$

Therefore

$$\Big\|x \mapsto \int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(L_\xi^m e^{tL_\xi}\check g(\xi))(x)d\xi\Big\|_{H^{l+1}(\mathbb{R})},$$
$$\Big\|x \mapsto \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(L_\xi^m e^{tL_\xi}\tilde\Pi(\xi)\check g(\xi))(x)d\xi\Big\|_{H^{l+1}(\mathbb{R})}$$
$$\lesssim e^{-\eta t}\|\xi \mapsto \ \|\check g(\xi)\|_{\dot H_\xi^{l+2m+1}([0,1])}\ \|_{L^2([-\pi,\pi])} \lesssim e^{-\eta t}\|g\|_{H^{l+2m+1}(\mathbb{R})}.$$

Since, for $2\le p\le\infty$, $H^{l+1}(\mathbb{R})$ is embedded in $W^{l,p}(\mathbb{R})$, this completes the proof of the proposition. □

Remark 3.4. We have included (3.16)(ii) to give a better account of large-time behavior of the phase. Yet, in our nonlinear stability analysis we use (3.16) only to ensure that the initial time layer remains localized, thus only for intermediate times $1/2\le t\le 1$. For this purpose, (3.16)(i) is sufficient. The proof of the latter estimate is easier to obtain and involves only frequency arguments in the spirit of the rest of the paper.

3.3. **Linear modulation bounds.** The above basic linear estimates are sufficient to control all terms in (3.12) and (3.13) in the localized case when $h_0\equiv 0$. To control the additional terms arising from the nonlocalized initial phase shift $h_0\bar U'$, we use the bounds in the following proposition. Here and elsewhere, we suppress the dependence of bounds on norms of the background periodic wave $\bar U$, the periodic right and left eigenbases ϕ_j and $\tilde\phi_j$, or other known periodic functions, which, by (H1), may be seen to be as smooth as needed for the $H^s([0,1])_{\rm per}$ bounds our arguments require.

For functions h with localized derivative, we will use repeatedly $\hat h(\xi)=\frac{1}{i\xi}\widehat{\partial_x h}(\xi)$. From now on, to make this possible, we will assume that all such functions are centered, in the sense that $h(-\infty)=-h(\infty)$ (which includes the case where h is itself localized and both terms vanish). In the statements of Theorems 1.10 and 1.12, it corresponds to the assumption that the global phase shift ψ_∞ is 0. Of course, in doing so, one does not lose in generality since this is achieved by replacing $\bar U$, h_0, ψ, *etc.* with $\bar U(\,\cdot\,-\psi_\infty)$, $h_0-\psi_\infty$, $\psi-\psi_\infty$, *etc.*

Proposition 3.5. *Under (H1)–(H3) and (D1)–(D3), for all $t\ge 0$, $2\le p\le\infty$, $l,m\ge 0$, $1\le j\le n+1$,*

$$\|\partial_x^l\partial_t^m s_j^{\rm p}(t)(h_0\bar U')\|_{L^p(\mathbb{R})}\lesssim (1+t)^{-\frac12(1-1/p)+\frac12-\frac{l+m}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})} \tag{3.23}$$

when $l+m\ge 1$,

$$\|\, s_j^{\rm p}(t)(h_0\bar U')\,\|_{L^\infty(\mathbb{R})}\ \lesssim\ \|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} \tag{3.24}$$

when ($l=m=0$ and $p=\infty$),

$$\|\, e_{n+1}\cdot s^{\rm p}(t)(h_0\bar U')-h_0\,\|_{L^p(\mathbb{R})}\ \lesssim\ (1+t^{\frac1p})\quad \|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} \tag{3.25}$$

when ($l=m=0$ and $2\le p\le\infty$), and, for $0\le l+2m\le K+1$,

$$\|\partial_x^l\partial_t^m\tilde S(t)(h_0\bar U')\|_{L^p(\mathbb{R})}\lesssim (1+t)^{-\frac12(1-1/p)}\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}(\mathbb{R})}. \tag{3.26}$$

Proof. [26] *(i) (Proof of* (3.23)*).* This follows applying the arguments of the proof of (3.14) in Proposition 3.3, once we have established the estimate

$$\|\xi\mapsto \alpha(\xi)^{1/2}|\langle\tilde\phi_j(\xi),(h_0\bar U')\check{}\,(\xi)\rangle_{L^2([0,1])}|\|_{L^\infty([-\pi,\pi])}$$
$$\lesssim\left(\sup_{|\xi|\le\xi_0}\|\partial_\xi\tilde\phi_j(\xi)\bar U'\|_{L^1([0,1])}+\sup_{|\xi|\le\xi_0}\|\tilde\phi_j(\xi)\bar U'\|_{L^2([0,1])}\right)\ \|\partial_x h_0\|_{L^1(\mathbb{R})}.$$

[26] Compare to the similar but much simpler argument of [25, Proposition 4.1], in the reaction-diffusion case.

The latter bound stems from first re-expressing

$$\alpha(\xi)^{1/2}\langle\tilde\phi_j(\xi),(h_0\bar U')\check{}\,(\xi)\rangle_{L^2([0,1])} = -i\,\alpha(\xi)^{1/2}\left\langle\frac{\tilde\phi_j(\xi)-\tilde\phi_j(0)}{\xi},\bar U'\right\rangle_{L^2([0,1])}\widehat{\partial_x h_0}(\xi)$$
$$+\sum_{j'\neq 0}\alpha(\xi)^{1/2}\frac{(\widehat{\tilde\phi_j\bar U'})(\xi,j')^*}{i(\xi+2\pi j')}\,\widehat{\partial_x h_0}(\xi+2j'\pi),$$

where $(\widehat{\tilde\phi_j\bar U'})(\xi,j')$ denotes the j'th Fourier coefficient in the Fourier expansion of periodic function $\tilde\phi_j(\xi)\bar U'$, then applying a Haussdorff-Young estimate, the Mean Value Theorem, Cauchy-Schwarz' inequality, and Parseval's identity.

(ii) (Proof of (3.24)*).* Thanks to (3.3), since $\tilde\phi_j(0)$ is constant equal to $\tilde\nu_j$, with the same kind of estimates one can bound in $L^\infty(\mathbb{R})$ the difference between $s_j^{\mathrm{p}}(t)(h_0\bar U')$ and

$$x\mapsto\int_{-\pi}^{\pi}e^{(-i\bar k a_j\xi-\bar k^2 b_j\xi^2)t}e^{i\xi x}\frac{1}{i\bar k\xi}\beta^{(j)}(0)[-i\langle\partial_\xi\tilde\phi_j(0),\bar U'\rangle_{L^2([0,1])}\widehat{\partial_x h_0}(\xi)+\tilde\nu_j\cdot\langle\bar U',[\check h_0(\xi)-\widehat{h_0}(\xi)]\rangle_{L^2([0,1])}]d\xi$$

by $C(1+t)^{-\frac12}\|\partial_x h_0\|_{L^1(\mathbb{R})}$. Now note that, for $\xi\in[-\pi,\pi]$

$$\begin{aligned}\langle\bar U',[\check h_0(\xi)-\widehat{h_0}(\xi)]\rangle_{L^2([0,1])} &= -\langle\bar U,\partial_x(\check h_0)(\xi)\rangle_{L^2([0,1])}\\ &= -\langle\bar U,(\partial_x\check{}h_0)(\xi)\rangle_{L^2([0,1])}+\langle\bar U\rangle\widehat{\partial_x h_0}(\xi)+i\xi\langle\bar U,[\check h_0(\xi)-\widehat{h_0}(\xi)]\rangle_{L^2([0,1])}\\ &= -\langle(\bar U\partial_x h_0)\check{}\,(\xi)\rangle+\langle\bar U\rangle\widehat{\partial_x h_0}(\xi)+i\xi\langle\bar U,[\check h_0(\xi)-\widehat{h_0}(\xi)]\rangle_{L^2([0,1])}\end{aligned}$$

leading to

$$\langle\bar U'[\check h_0(\xi)-\widehat{h_0}(\xi)]\rangle = -[(\bar U-\langle\bar U\rangle)\partial_x h_0]\hat{}\,(\xi)+i\xi\langle\bar U,[\check h_0(\xi)-\widehat{h_0}(\xi)]\rangle_{L^2([0,1])}. \tag{3.27}$$

Again the extra ξ factor makes the contribution of the last term negligible so that we are left with proving a $L^\infty(\mathbb{R})$ bound on

$$x\mapsto\int_{-\pi}^{\pi}e^{(-i\bar k a_j\xi-\bar k^2 b_j\xi^2)t}e^{i\xi x}\frac{1}{i\bar k\xi}\beta^{(j)}(0)[-i\langle\partial_\xi\tilde\phi_j(0),\bar U'\rangle_{L^2([0,1])}\partial_x h_0-\tilde\nu_j\cdot(\bar U-\langle\bar U\rangle)\partial_x h_0]\hat{}\,(\xi)d\xi.$$

Then the proof of (3.24) is achieved as was the one of (3.15), writing the main contribution as a term exponentially-decaying in time plus a convolution with an explicit errorfunction.[27]

(iii) (Proof of (3.25)*).* The $L^\infty(\mathbb{R})$ bound follows from (3.24) and $\|h_0\|_{L^\infty(\mathbb{R})}\leq\|\partial_x h_0\|_{L^1(\mathbb{R})}$. By interpolation we only need now the $L^2(\mathbb{R})$ bound.

Since $\|\ \xi\longmapsto|\xi|^{-1}\ \|_{L^2(\mathbb{R}\setminus[-\pi,\pi])}\lesssim 1$,

$$\left\|\ x\longmapsto h_0(x)-\int_{-\pi}^{\pi}e^{i\xi x}\quad\widehat{h_0}(\xi)d\xi\ \right\|_{L^2(\mathbb{R})}\lesssim\|\partial_x h_0\|_{L^1(\mathbb{R})}.$$

[27]Up to the explicit computation of the final convolution kernel, the arguments of the proofs of (3.15)-(3.16) and (3.24)-(3.25) are the ones refined to obtain (4.24).

Combing this with arguments of the proof of (3.24), we bound in $L^2(\mathbb{R})$ the difference between $e_{n+1}\cdot s^{\mathrm{p}}(t)(h_0\bar{U}')-h_0$ and

$$x\mapsto\int_{-\pi}^{\pi}e^{i\xi x}\left[\sum_{j=1}^{n+1}e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}\frac{1}{i\bar{k}}e_{n+1}\cdot\beta^{(j)}(0)\langle\partial_\xi\tilde{\phi}_j(0),\bar{U}'\rangle_{L^2([0,1])}\widehat{h_0}(\xi)-\widehat{h_0}(\xi)\right]d\xi$$
$$-\int_{-\pi}^{\pi}e^{i\xi x}\sum_{j=1}^{n+1}e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}\frac{1}{i\bar{k}\xi}e_{n+1}\cdot\beta^{(j)}(0)\tilde{\nu}_j\cdot[(\bar{U}-\langle\bar{U}\rangle)\partial_x h_0]^{\wedge}(\xi)d\xi$$

by $C\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}$. In the sum the latter term is bounded following the proof of (3.16)(ii). To bound the first term, we first observe that, thanks to (3.6),

$$\langle\partial_\xi\tilde{\phi}_j(0),\bar{U}'\rangle_{L^2([0,1])}\;=\;\langle\partial_\xi\tilde{\phi}_j(0),q_{n+1}(0)\rangle_{L^2([0,1])}\;=\;i\bar{k}\tilde{\beta}^{(j)}_{n+1}(0)\;=\;i\bar{k}\tilde{\beta}^{(j)}(0)\cdot e_{n+1}$$

therefore

$$\frac{1}{i\bar{k}}\sum_{j=1}^{n+1}e_{n+1}\cdot\beta^{(j)}(0)\langle\partial_\xi\tilde{\phi}_j(0),\bar{U}'\rangle_{L^2([0,1])}\;=\;e_{n+1}^T\sum_{j=1}^{n+1}\beta^{(j)}(0)\tilde{\beta}^{(j)}(0)^Te_{n+1}\;=\;e_{n+1}\cdot e_{n+1}\;=\;1.$$

Then the function to bound is written

$$x\mapsto\int_{-\pi}^{\pi}e^{i\xi x}\sum_{j=1}^{n+1}\frac{e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}-1}{i\bar{k}\xi}\beta^{(j)}_{n+1}(0)\tilde{\beta}^{(j)}_{n+1}(0)\;\bar{k}\widehat{\partial_x h_0}(\xi)d\xi$$

and is bounded in $L^2(\mathbb{R})$ splitting

$$\frac{e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}-1}{i\bar{k}\xi}\;=\;\frac{e^{(-i\bar{k}a_j\xi-\bar{k}^2b_j\xi^2)t}-e^{-\bar{k}^2b_j\xi^2t}}{i\bar{k}\xi}+\frac{e^{-\bar{k}^2b_j\xi^2t}-1}{i\bar{k}\xi}$$

and using both $\left|\frac{e^{-\bar{k}^2b_j\xi^2t}-1}{i\bar{k}\xi}\right|\lesssim\sqrt{t}$ and the arguments of the proof of (3.24)(ii) where I_t^ε is replaced with

$$\tilde{I}_t^\varepsilon\;:=\;\left[\;t\left(\min_{1\leq j\leq n+1}(a_j)_--\varepsilon\right)\;,\;t\left(\max_{1\leq j\leq n+1}(a_j)_++\varepsilon\right)\;\right]$$

($_-$ and $_+$ denoting negative and positive parts).

(iv) (Proof of (3.26)*).* The last part of (3.9) is bounded as was s^{p}_j, with the usual improvement in integrability and decay coming from the extra ξ factor. Once we have proved

$$\|\xi\mapsto\;\|\bar{U}'[\check{h}_0(\xi)-\widehat{h_0}(\xi)]\|_{\dot{H}^{l+2m+1}_\xi([0,1]))}\;\|_{L^2([-\pi,\pi])}\lesssim\|\partial_x h_0\|_{H^{l+2m+1}(\mathbb{R})},$$

the contribution of the high frequencies of h_0 to the remaining terms is estimated following the proof of (3.18). Since, for $\xi\in[-\pi,\pi]$ there holds

$$\begin{aligned}\|\bar{U}'[\check{h}_0(\xi)-\widehat{h_0}(\xi)]\|_{\dot{H}^{l+2m+1}_\xi([0,1]))}&\leq\sum_{j'\neq0}\|e^{2\pi j'\cdot}\;\bar{U}'\|_{\dot{H}^{l+2m+1}_\xi([0,1]))}|\widehat{h_0}(\xi+2\pi j')|\\&\lesssim\left(1+\|\bar{U}'\|_{H^{l+2m+1}([0,1]))}\right)\sum_{j'\neq0}|\xi+2\pi j'|^{l+2m+1}|\widehat{h_0}(\xi+2\pi j')|\\&\lesssim\sum_{j'\neq0}\frac{1}{|\xi+2\pi j'|}\left|(\widehat{\partial_x^{l+2m+2}h_0})(\xi+2\pi j')\right|\\&\lesssim\sqrt{\sum_{j'\neq0}\left|(\widehat{\partial_x^{l+2m+2}h_0})(\xi+2\pi j')\right|^2},\end{aligned}$$

by squaring and integrating we obtain the needed bound.

To deal with low-frequency contributions, we use the following refinements:

$$|e^{tL_\xi}(1-\alpha(\xi))|_{\dot H^{l+2m+1}_\xi([0,1])\to \dot H^{l+2m+1}_\xi([0,1])} \lesssim |\xi| e^{-\eta t}$$
$$|\alpha(\xi)^{1/2}(\tilde\Pi(\xi)-\tilde\Pi(0))|_{\dot H^{l+2m+1}_\xi([0,1])\to \dot H^{l+2m+1}_\xi([0,1])} \lesssim |\xi|$$

(for some $\eta>0$ and all $\xi\in[-\pi,\pi]$). Then, since $\tilde\Pi(0)\bar U'=0$,

$$\alpha(\xi)^{1/2}\,\tilde\Pi(\xi)\bar U' \;=\; \alpha(\xi)^{1/2}\,\tilde\Pi(\xi)(\tilde\Pi(\xi)-\tilde\Pi(0))\bar U',$$

and, by following the proof of (3.18), we reduce the bound on the remaining terms to

$$\|\xi\mapsto\ \|\bar U'\|_{\dot H^{l+2m+1}_\xi([0,1]))}\ |\xi|\,|\widehat{h_0}(\xi)|\ \|_{L^2([-\pi,\pi])} \lesssim \|\partial_x h_0\|_{L^2(\mathbb{R})}.$$

□

Remark 3.6. Unlike what occurs in the linearly decoupled case, linear bounds for a localized initial datum $d_0\in L^1(\mathbb{R})$ or a nonlocalized one of shift type $h_0\bar U'$, $\partial_x h_0\in L^1(\mathbb{R})$ provide in general the same decay rates. As a consequence, once these bounds are proved, the proof of nonlinear stability is identical to the one for localized perturbations [30]. In particular, the slow decay rate due to the Jordan block is compensated for nonlinear terms by the fact that they come in flux form.

3.4. **Nonlinear stability.** From differential equation (2.6) together with integral equation (3.12)-(3.13), we readily obtain short-time existence, uniqueness and continuity with respect to t of solution $(v,\psi_t,\psi_x)\in H^K(\mathbb{R})\times H^{K-1}(\mathbb{R})\times H^K(\mathbb{R})$ by a standard contraction-mapping argument treating most of the terms as sources in a heat equation. Associated with this solution define so long as it is finite

$$\zeta(t) := \sup_{0\le s\le t}\|(v,\psi_t,\psi_x)(s)\|_{H^K(\mathbb{R})\times H^{K-1}(\mathbb{R})\times H^K(\mathbb{R})}(1+s)^{1/4}. \tag{3.28}$$

Combining linear estimates with Proposition 2.5, we now prove an inequality for ζ that will yield global existence of our solutions.

Lemma 3.7. *Under assumptions (H1)–(H3) and (D1)–(D3), there exist positive constants C and ε such that if $(d_0,\partial_x h_0)$ is such that for some $T>0$*

$$E_0 := \|(d_0,\partial_x h_0)\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})}\ \le\ \varepsilon \qquad \text{and} \qquad \zeta(T)\ \le\ \varepsilon$$

then, for all $0\le t\le T$,

$$\zeta(t)\le C(E_0+\zeta(t)^2).$$

Proof. [28] By (2.7)–(2.9) and corresponding bounds on the derivatives together with definition (3.28) and equation (2.14) (used to bound v_t),

$$\|\mathcal{N}(t)\|_{L^1(\mathbb{R})\cap H^1(\mathbb{R})} \lesssim \|(v,v_x,v_{xx},\psi_t,\psi_x,\psi_{xx})(t)\|^2_{H^1(\mathbb{R})} \le C\zeta(t)^2(1+t)^{-\frac12},$$

$$\|(\mathcal{Q},\mathcal{R},\mathcal{S})(t)\|_{L^1(\mathbb{R})\cap H^1(\mathbb{R})} \lesssim \|(v,v_x,\psi_t,\psi_x)(t)\|^2_{H^1(\mathbb{R})} \le C\zeta(t)^2(1+t)^{-\frac12}, \tag{3.29}$$

so long as $\zeta(t)$ remains small. Applying the bounds (3.14)(i)–(3.19)(i) and (3.23)–(3.26) of Propositions 3.3 and 3.5 to system (3.12)-(3.13), we obtain for any $2\le p<\infty$

$$\begin{aligned}\|v(t)\|_{L^p(\mathbb{R})} &\le C(1+t)^{-\frac12(1-1/p)}E_0 + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac12(1-1/p)-\frac12}(1+s)^{-\frac12}ds\\ &\le C_p(E_0+\zeta(t)^2)(1+t)^{-\frac12(1-1/p)}\end{aligned} \tag{3.30}$$

[28]Compare to the argument of [29, Lemma 4.2], regarding localized perturbations in the decoupled case.

and, with

$$\int_0^t s^{\mathrm{p}}(t-s)\partial_t\mathcal{S}(s)ds \;=\; -\int_0^t \partial_t[s^{\mathrm{p}}](t-s)\mathcal{S}(s)ds + s^{\mathrm{p}}(0)\mathcal{S}(t) - s^{\mathrm{p}}(t)\mathcal{S}(0), \tag{3.31}$$

$$\begin{aligned}\|(\psi_t,\psi_x,\psi_{xx})(t)\|_{W^{K-1,p}(\mathbb{R})} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}(E_0+\zeta(t)^2)\\ &\quad + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac{1}{2}(1-1/p)-1/2}(1+s)^{-\frac{1}{2}}ds\\ &\le C_p(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)}.\end{aligned} \tag{3.32}$$

Since the hypotheses of Proposition 2.5 are verified, from (2.13) and (3.30)–(3.32), we thus obtain

$$\|v(t)\|_{H^K(\mathbb{R})} \le C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{4}}. \tag{3.33}$$

Combining this with (3.32) for $p=2$, we obtain the result. □

We are now ready to prove Theorem 1.10.

Proof of Theorem 1.10. As already mentioned, short-time existence and uniqueness ensuring continuously in time $(v,\psi_t,\psi_x)\in H^K(\mathbb{R})\times H^{K-1}(\mathbb{R})\times H^K(\mathbb{R})$ are proved in a standard way. Therefore, by Lemma 3.7 it follows by continuous induction that solutions are global in time and satisfy $\zeta(t)\le 2CE_0$ for $t\ge 0$, if $E_0\le \min(\{1/4C^2,\varepsilon,\varepsilon/2C\})$, yielding by (3.28) the result (1.13) for $p=2$.

For any $p_*<\infty$, applying (3.30)–(3.32), we obtain (1.13) for $2\le p\le p_*$ with a uniform constant C. Now rewrite (2.8) as $\mathcal{R}=\mathcal{R}_1+\partial_x\mathcal{R}_2$ with

$$\mathcal{R}_1 := -v\psi_t - \bar{k}^2 v\left(\frac{\psi_x}{1-\psi_x}\right)_x + \bar{k}^2\bar{U}_x\frac{\psi_x^2}{1-\psi_x}, \qquad \mathcal{R}_2 := \bar{k}^2 v\frac{\psi_x}{1-\psi_x}.$$

Taking $p_*\ge 4$ and estimating

$$\|(\mathcal{Q},\mathcal{R}_1,\mathcal{R}_2,\mathcal{S})(t)\|_{L^2(\mathbb{R})} \lesssim \|(v,\psi_t,\psi_x,\psi_{xx})(t)\|_{L^4(\mathbb{R})}^2 \le CE_0(1+t)^{-\frac{3}{4}} \tag{3.34}$$

in place of the weaker (3.29), then applying to integral terms (3.17)(ii) in place of (3.17)(i), we obtain, bounding again the $\partial_t\mathcal{S}$ contribution using (3.31),

$$\begin{aligned}\|(\psi_t,\psi_x,\psi_{xx})(t)\|_{W^{K-1,p}(\mathbb{R})} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 + CE_0^2\int_0^t (1+t-s)^{-\frac{1}{2}(1/2-1/p)-1/2}(1+s)^{-\frac{3}{4}}ds\\ &\le CE_0(1+t)^{-\frac{1}{2}(1-1/p)},\end{aligned} \tag{3.35}$$

for $2\le p\le\infty$. Likewise, using (3.34) together with bound

$$\|(\mathcal{Q},\mathcal{R}_1)(t)\|_{H^{K-1}(\mathbb{R})} + \|(\mathcal{R}_2,\mathcal{S})(t)\|_{H^K(\mathbb{R})} \lesssim E_0^2(1+t)^{-\frac{1}{2}}$$

obtained from the bound on ζ, and

$$\int_0^t \tilde{S}(t-s)\partial_t\mathcal{S}(s)ds \;=\; -\int_0^t \partial_t[\tilde{S}](t-s)\mathcal{S}(s)ds + \tilde{S}(0)\mathcal{S}(t) - \tilde{S}(t)\mathcal{S}(0),$$

instead of (3.31), we may use (3.19)(ii) rather than (3.19)(i) to get, provided $l+2m\le K-3$,

$$\begin{aligned}\|\partial_x^l\partial_t^m v(t)\|_{L^p(\mathbb{R})} &\le C\,(1+t)^{-\frac{1}{2}(1-1/p)}E_0 + C\,E_0^2\int_0^t e^{-\eta(t-s)}(1+s)^{-\frac{1}{2}}ds\\ &\quad + C\,E_0^2\int_0^t (1+t-s)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}}(1+s)^{-\frac{3}{4}}ds\\ &\le C\,E_0(1+t)^{-\frac{1}{2}(1-1/p)}\end{aligned} \tag{3.36}$$

and achieve the proof of (1.13) for $2 \le p \le \infty$.

Estimate (1.14) then follows through (3.12) using (3.17)(i) and (3.23), by

$$\|\psi(t)\|_{L^\infty(\mathbb{R})} \ \le\ CE_0 + CE_0^2 \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}} ds \ \le\ CE_0,$$

yielding nonlinear stability in L^∞ since

$$\tilde{u}(x,t) - \bar{U}(x) = \tilde{u}(x,t) - \bar{U}(x+\psi(x,t)) + \bar{U}(x+\psi(x,t),t) - \bar{U}(x)$$

so that, by Lemma 2.7,

$$\|\tilde{u}(t) - \bar{U}\|_{L^\infty(\mathbb{R})} \lesssim \|v(t)\|_{L^\infty(\mathbb{R})} + \|\psi(t)\|_{L^\infty(\mathbb{R})}\|\psi_x(t)\|_{L^\infty(\mathbb{R})} + \|\psi(t)\|_{L^\infty(\mathbb{R})} \sup_{[0,1]} |\bar{U}'| \ .$$

Finally, Lemma 2.7 provides (1.15). □

In proving Theorem 1.10, we actually got or could get more estimates than announced. Since we need these extra estimates to prove Theorem 1.12, we record here for later these other bounds.

Proposition 3.8. *Under the assumptions of Theorem 1.10 and with its notations, for* $2 \le p \le \infty$ *and* $t \ge 0$*,*

$$\begin{aligned} \|v(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)} \\ \|\partial_x^l \partial_t^m v(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}} \\ \|v(t)\|_{H^K(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{4}}, \end{aligned} \tag{3.37}$$

when $1 \le l+2m \le K-3$ *and when* $3 \le l+m$ *and* $l+2m \le K+1$

$$\begin{aligned} \|(\psi_t,\psi_x)(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)} \\ \|(\psi_{tt},\psi_{tx},\psi_{xx})(t)\|_{L^p(\mathbb{R})} &\lesssim E_0 \ln(2+t)(1+t)^{-\frac{3}{4}} \\ \|\partial_x^l \partial_t^m \psi(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{3}{4}}, \end{aligned} \tag{3.38}$$

where v *is as in* (2.5)*.*

Proof. Estimate (3.37)(i) is (1.13)(i) and (3.37)(iii) follows from the bound on ζ. Estimate (3.37)(ii) is proved incorporating extra decay obtained from differentiation in (3.36).

Estimate (3.38)(i) is (1.13)(ii). Bounds (3.38)(ii)-(iii) follow along the lines of (3.35) benefiting from extra decay provided by differentiation. □

4. Behavior

The first step in going from Theorem 1.10 to Theorem 1.12 is to obtain a refined expansion of the solution operator $S(t) = e^{tL}$, allowing us to split further the integral equation (3.11). This proceeds by refining the spectral expansions from which they ultimately derive.

4.1. Refined spectral expansion and decompositions.

Lemma 4.1 ([48, 49])**.** *Assuming (H1)–(H3), we may choose a parametrization in such a way that, for the quantities involved in* (3.5)*,*

$$\partial_\xi q_{n+1}(0) = i\bar{k}\partial_k U_{|(\bar{M},\bar{k})}. \tag{4.1}$$

Proof. We have already observed that the proof of Proposition 1.7 provides

$$L_0(\partial_\xi q_{n+1}(0) - i\bar{k}\partial_k U_{|(\bar{M},\bar{k})}) \in \Sigma_0.$$

The latter point implies that $\partial_\xi q_{n+1}(0) - i\bar{k}\partial_k U_{|(\bar{M},\bar{k})} \in \Sigma_0$. Therefore, we only need to show that, for $1 \le j \le n+1$, $\langle q_j(0), \partial_\xi q_{n+1}(0)\rangle = \langle q_j(0), i\bar{k}\partial_k U_{|(\bar{M},\bar{k})}\rangle$.

Moreover, the proof of Proposition 1.7 also yields $\langle q_j(0), \partial_\xi q_{n+1}(0)\rangle = 0$ for all $j \ne n+1$. We may assume that this relation holds for $j = n+1$ by normalizing $q_{n+1}(\xi)$ according to $\langle \tilde{q}_{n+1}(0), q_{n+1}(\xi)\rangle = 1$. Indeed, once this is done, expanding to first order in ξ this normalization and the duality relation $\langle \tilde{q}_{n+1}(\xi), q_{n+1}(\xi)\rangle = 1$ provides the desired cancellation.

Now from the second part of (A.3)(iii), we already know $\langle q_j(0), i\bar{k}\partial_k U_{|(\bar{M},\bar{k})}\rangle = 0$ for all $j \ne (n+1)$. Moreover, up to changing parametrization by a k-dependent shift, we may add to $\partial_k U_{|(\bar{M},\bar{k})}$ a suitable multiple of $\bar{U}'$ and get

$$\langle \tilde{q}_{n+1}(0), i\bar{k}\partial_k U_{|(\bar{M},\bar{k})}\rangle \;=\; 0. \tag{4.2}$$

This yields (4.1). $\square$

Accordingly, following [26], we refine (3.7) and re-express $S(t)$ as

$$S(t) = R^{\mathrm{p}}(t) + R^M(t) + \tilde{R}(t), \tag{4.3}$$

where

$$R^{\mathrm{p}}(t) = (\bar{U}' + \partial_k\bar{U}\ \bar{k}\partial_x)\, e_{n+1}\cdot s^{\mathrm{p}}(t), \tag{4.4}$$

with s^{p} as in (3.8),

$$R^M(t) := \partial_M\bar{U}\cdot s^M(t), \qquad s^M(t) := \sum_{j=1}^{n+1} s_j^M(t), \qquad s_j^M(t) := \begin{pmatrix} & 0 \\ \mathrm{Id}_n & \vdots \\ & 0 \end{pmatrix} \bar{k}\partial_x s_j^{\mathrm{p}}(t), \tag{4.5}$$

and

$$\begin{aligned}(\tilde{R}(t)g)(x) := &\int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(e^{tL_\xi}\check{g}(\xi))(x)d\xi + \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(e^{tL_\xi}\tilde{\Pi}(\xi)\check{g}(\xi))(x)d\xi \\ &+ \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi) \quad \sum_{j=1}^{n+1} e^{\lambda_j(\xi)t}\ \frac{\phi_j^{quad}(\xi,x)}{i\bar{k}\xi}\ \langle\tilde{\phi}_j(\xi), \check{g}(\xi)\rangle_{L^2([0,1])}\ d\xi,\end{aligned} \tag{4.6}$$

with

$$\phi_j^{quad}(\xi,x) \;=\; \phi_j(\xi,x) - (i\bar{k}\xi)\sum_{l=1}^{n}\beta_l^{(j)}(\xi)q_l(0,x) - \beta_{n+1}^{(j)}(\xi)\big(q_{n+1}(0,x) + \xi\partial_\xi q_{n+1}(0,x)\big),$$

where α is an already introduced smooth cutoff supported where $|\xi| \le \xi_0$, $\tilde{\Pi}(\xi)$ is, as defined in (3.10), the complementary projection of $\Pi^{\mathrm{p}}(\xi)$, the spectral projection on $\Sigma_\xi := \mathrm{Span}\{\phi_j(\xi)\}_{j=1}^{n+1}$.

With (4.4)-(4.5) in mind, for $\tilde{u}$ satisfying (1.1), we may refine the nonlinear decomposition (2.5) into

$$\begin{aligned} z(x,t) &= \tilde{u}(x-\psi(x,t),t) - \bar{U}(x) - \partial_k\bar{U}(x)\,\bar{k}\psi_x(x,t) - \partial_M\bar{U}(x)\cdot M(x,t) \\ &= v(x,t) - \partial_k\bar{U}(x)\,\bar{k}\psi_x(x,t) - \partial_M\bar{U}(x)\cdot M(x,t)\end{aligned} \tag{4.7}$$

where $\psi(x,t)$ still satisfies (3.12), and $M(x,t)$ is defined through (4.8)(iii) just below. Further, recall $d_0 := \tilde{u}_0(\cdot - h_0(\cdot)) - \bar{U}$, and let χ be the smooth cutoff function of (3.12)–(3.13), with $\chi(t) = 0$ for $t \le 1/2$, and $\chi(t) = 1$ for $t \ge 1$. With definitions (4.3)–(4.5), we have the following lemma.

Lemma 4.2. *For $\mathcal{N}$ as in* (2.6)–(2.9)*, the nonlinear residual z defined in* (4.7) *satisfies*

$$\begin{aligned}
z(t) &= \tilde{R}(t)(d_0 + h_0\bar{U}') + \int_0^t \tilde{R}(t-s)\mathcal{N}(s)ds \\
&\quad + (1-\chi(t))\Big(R^{\mathrm{p}}(t)(d_0 + h_0\bar{U}') - h_0\bar{U}' - \partial_k\bar{U}\,\bar{k}\partial_x h_0 + \int_0^t R^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\Big), \\
\psi(t) &= e_{n+1}\cdot s^{\mathrm{p}}(t)(d_0 + h_0\bar{U}') + \int_0^t e_{n+1}\cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds \\
&\quad - (1-\chi(t))\Big(e_{n+1}\cdot s^{\mathrm{p}}(t)(d_0 + h_0\bar{U}') - h_0 + \int_0^t e_{n+1}\cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\Big), \\
M(t) &= s^M(t)(d_0 + h_0\bar{U}') + \int_0^t s^M(t-s)\mathcal{N}(s)ds.
\end{aligned} \tag{4.8}$$

Proof. Recall, (3.12)–(3.13), that we have chosen ψ so that ψ and v satisfy both (4.8)(ii) and

$$\begin{aligned}
v(t) &= \tilde{S}(t)(d_0 + h_0\bar{U}') + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds \\
&\quad + (1-\chi(t))\Big(\bar{U}'e_{n+1}\cdot s^{\mathrm{p}}(t)(d_0 + h_0\bar{U}') - h_0\bar{U}' + \int_0^t \bar{U}'e_{n+1}\cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)ds\Big),
\end{aligned} \tag{4.9}$$

where $\tilde{S}(t) = S(t) - \bar{U}'e_{n+1}\cdot s^{\mathrm{p}}(t)$. Using now $z(t) = v(t) - \partial_k\bar{U}\,\bar{k}\psi_x(t) - \partial_M\bar{U}\cdot M(t)$,

$$\tilde{R}(t) = \tilde{S}(t) - \partial_k\bar{U}\,\bar{k}\partial_x e_{n+1}\cdot s^{\mathrm{p}}(t) - \partial_M\bar{U}\cdot s^M(t),$$

and $R^{\mathrm{p}}(t) = (\bar{U}' + \partial_k\bar{U}\,\bar{k}\partial_x)\,e_{n+1}\cdot s^{\mathrm{p}}(t)$, $R^M(t) = \partial_M\bar{U}\cdot s^M(t)$, equation (4.8)(i) follows from (4.8)(ii)-(iii) and (4.9). □

We now establish the refined linear bounds needed to estimate the terms involved in (4.8)

4.2. **Refined linear stability estimates.**

4.2.1. *Refined basic estimates.*

Proposition 4.3. *Under assumptions (H1)-(H3) and (D1)-(D3), for all $t \geq 0$, $1 \leq j \leq n$, $2 \leq p \leq \infty$, and for some $\eta > 0$, $0 \leq l + 2m \leq K+1$, and $2 \leq p \leq \infty$,*

$$\Big\|\partial_x^l\partial_t^m s_j^M(t)\,g\Big\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac{l+m}{2}}\|g\|_{L^1(\mathbb{R})}\\(1+t)^{-\frac12(1/2-1/p)-\frac{l+m}{2}}\|g\|_{L^2(\mathbb{R})}\end{cases}$$

$$\Big\|\partial_x^l\partial_t^m\tilde{R}(t)g\Big\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}e^{-\eta t}\|g\|_{H^{l+2m+1}(\mathbb{R})} + (1+t)^{-\frac12(1-1/p)-\frac12}\|g\|_{L^1(\mathbb{R})}\\e^{-\eta t}\|g\|_{H^{l+2m+1}(\mathbb{R})} + (1+t)^{-\frac12(1/2-1/p)-\frac12}\|g\|_{L^2(\mathbb{R})}\end{cases} \tag{4.10}$$

and, for $1 \leq r \leq K+1$,

$$\Big\|\partial_x^l\partial_t^m s_j^M(t)\partial_x^r g\Big\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac{l+m}{2}-\frac12}\|g\|_{L^1(\mathbb{R})}\\(1+t)^{-\frac12(1/2-1/p)-\frac{l+m}{2}-\frac12}\|g\|_{L^2(\mathbb{R})}\end{cases} \tag{4.11}$$

$$\Big\|\partial_x^l\partial_t^m\tilde{R}(t)\partial_x^r g\Big\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})} + (1+t)^{-\frac12(1-1/p)-1}\|g\|_{L^1(\mathbb{R})}\\e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})} + (1+t)^{-\frac12(1/2-1/p)-1}\|g\|_{L^2(\mathbb{R})}\end{cases}. \tag{4.12}$$

Proof. Bounds on s_j^M follow directly from known bounds on s_j^{p}, whereas the proofs of estimates on $\tilde{R}$ are completely similar to the proofs of bounds on $\tilde{S}$ in Proposition 3.3, with extra decay coming from a higher-order expansion of $\phi_j(\xi)$, leading to an extra ξ factor in the third term of (4.6). □

4.2.2. *Refined linear modulation bounds.*

Proposition 4.4. *Under assumptions (H1)–(H3) and (D1)–(D3), for all $t \geq 0$, $2 \leq p \leq \infty$, and $0 \leq l + 2m \leq K + 1$,*

$$\|\partial_x^l \partial_t^m s_j^M(t)(h_0 \bar{U}')\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})}, \tag{4.13}$$

$$\|\partial_x^l \partial_t^m \tilde{R}(t)(h_0 \bar{U}')\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R}) \cap H^{l+2m+1}(\mathbb{R})}. \tag{4.14}$$

Proof. Again, bounds on s_j^M follow directly from known bounds on s_j^{p}, whereas the proof of estimates on $\tilde{R}$ is completely similar to the proof of bounds on $\tilde{S}$ in Proposition 4.4, with extra decay coming from the higher-order expansion of $\phi_j(\xi)$, leading to an extra ξ factor in the third term of (4.6). $\square$

4.3. **Refined nonlinear stability estimates.** With these preparations, we obtain the following refinement of Theorem 1.10.

Proposition 4.5. *Under the assumptions of Theorem 1.10, for all $t \geq 0$, $2 \leq p \leq \infty$,*

$$\begin{aligned} \|z(t)\|_{L^p(\mathbb{R})} &\lesssim E_0 \ln(2+t)\ (1+t)^{-\frac{3}{4}} \\ \|\partial_x^l \partial_t^m z(t)\|_{L^p(\mathbb{R})} &\lesssim E_0 (1+t)^{-\frac{3}{4}}, \qquad 1 \leq l + 2m \leq K - 3 \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \|M(t)\|_{L^p(\mathbb{R})} &\leq C E_0 (1+t)^{-\frac{1}{2}(1-1/p)} \\ \|(M_t, M_x)(t)\|_{L^p(\mathbb{R})} &\leq C E_0 \ln(2+t)(1+t)^{-\frac{3}{4}} \\ \|\partial_x^l \partial_t^m M(t)\|_{L^p(\mathbb{R})} &\leq C E_0 (1+t)^{-\frac{3}{4}}, \qquad l + m \geq 2,\ l + 2m \leq K + 1. \end{aligned} \tag{4.16}$$

Proof. By (3.38) and (3.37), we find

$$\|\mathcal{N}(t)\|_{H^{K-2}(\mathbb{R})} \leq C E_0 (1+t)^{-\frac{3}{4}}.$$

Applying bounds (4.10)(i)–(4.12)(ii) and (4.13)-(4.14)(i) of Propositions 4.3 and 4.4 to the system (4.8), we obtain for any $2 \leq p \leq \infty$,

$$\begin{aligned} \|z(t)\|_{L^p(\mathbb{R})} &\leq C E_0 (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} + C E_0 \int_0^t (1+t-s)^{-\frac{1}{2}(1/2-1/p)-1} (1+s)^{-\frac{3}{4}} ds \\ &\leq C E_0 \ln(2+t)\ (1+t)^{-\frac{3}{4}}, \end{aligned}$$

verifying (4.15)(i). Estimate (4.15)(ii) is proved in the same way dropping the log term thanks to an extra decay of $(1+t-s)^{-1/2}$ in the integral. Bound (4.16) follows similarly as for the previous bounds on ψ_x, ψ_t. $\square$

This verifies the estimate (1.16), thus validating the *ansatz* (1.6). It now remains to establish comparisons with solutions of (1.7). Note that, as explained in Appendix B, taking into account simplifications due to asymptotic equivalence of quadratic approximants and the change of variables $(\mathrm{Id} - \psi(\cdot, t))^{-1}$, it is sufficient to prove comparisons with solutions to system (B.36), which we do below.

4.4. **Quadratic approximation.** Before entering into comparisons with the Whitham equations, we identify now the main part of the nonlinear terms. To this end, we first need estimates revealing the characteristic speeds of each of the linear parts. For this purpose, we introduce for a_j as in (3.3)

$$D_j \;=\; \partial_t + a_j\;\bar{k}\partial_x\;.$$

Then setting

$$V_j = \beta^{(j)}(0), \qquad \tilde{V}_j = \tilde{\beta}^{(j)}(0), \tag{4.17}$$

we define the following total derivatives

$$D \;=\; \sum_{j=1}^{n} V_j\, D_j\, \tilde{V}_j^T\;.$$

Note that, more compactly, we actually have

$$D \;=\; \partial_t + \left(\begin{pmatrix} dF_{|(\bar{M},\bar{k})} \\ -d\omega_{|(\bar{M},\bar{k})}\end{pmatrix} - \bar{c}\mathrm{Id}\right)\bar{k}\partial_x \;\;=\; \partial_t + A_*\bar{k}\partial_x.$$

Lemma 4.6. *Assuming (H1)-(H3) and (D1)-(D3), for all $t\geq 0$, $2\leq p\leq\infty$,*

$$\left\|\partial_x^l\partial_t^m D s^{\mathrm{p}}(t)g\right\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}-\frac{1}{2}}\|g\|_{L^1(\mathbb{R})}\\(1+t)^{-\frac{1}{2}(1/2-1/p)-\frac{l+m}{2}-\frac{1}{2}}\|g\|_{L^2(\mathbb{R})}\end{cases},$$

$$\left\|\partial_x^l\partial_t^m D s^{\mathrm{p}}(t)(\partial_x^r g)\right\|_{L^p(\mathbb{R})} \lesssim \min\begin{cases}(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}-1}\|g\|_{L^1(\mathbb{R})}\\(1+t)^{-\frac{1}{2}(1/2-1/p)-\frac{l+m}{2}-1}\|g\|_{L^2(\mathbb{R})}\end{cases}$$

when $1\leq r\leq K+1$,

$$\left\|\partial_x^l\partial_t^m D s^{\mathrm{p}}(t)(h_0\bar{U}')\right\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}-\frac{1}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})}, \tag{4.18}$$

and, for g periodic of period 1,

$$\left\|\partial_x^l\partial_t^m D s^{\mathrm{p}}(t)(h_0 g)\right\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})}. \tag{4.19}$$

Proof. From the observation that

$$\begin{aligned}(\partial_x^l\partial_t^m D s^{\mathrm{p}}(t)g)(x) &= \sum_{j=1}^{n+1} V_j\int_{-\pi}^{\pi}\alpha(\xi)e^{\lambda_j(\xi)t}e^{i\xi x}\frac{\lambda_j(\xi)+i\xi a_j}{i\bar{k}\xi}(i\xi)^l\lambda_j(\xi)^m\langle\tilde{\phi}_j(\xi),\check{g}(\xi)\rangle d\xi\\ &+\sum_{j,j'=1}^{n+1} V_j\int_{-\pi}^{\pi}\alpha(\xi)e^{\lambda_{j'}(\xi)t}e^{i\xi x}\frac{\lambda_{j'}(\xi)+i\xi a_j}{i\bar{k}\xi}(i\xi)^l\lambda_j(\xi)^m\,\tilde{V}_j\cdot(\beta^{(j')}(\xi)-\beta^{(j')}(0))\,\langle\tilde{\phi}_{j'}(\xi),\check{g}(\xi)\rangle d\xi\end{aligned} \tag{4.20}$$

with for all j,j'

$$\lambda_j(\xi)+i\xi a_j = \mathcal{O}(\xi^2), \qquad (\lambda_{j'}(\xi)+i\xi a_j)\tilde{V}_j\cdot(\beta^{(j')}(\xi)-\beta^{(j')}(0)) = \mathcal{O}(\xi^2),$$

the proof follows the lines of previous estimates on s_j^{p}. In particular, the discrepancy between decay rates in (4.18) and (4.19) is due to the fact that (4.18) benefits from cancellations

$$\langle\tilde{\phi}_j(0),\bar{U}'\rangle \;=\; 0, \qquad \text{for all }\; 1\leq j\leq n+1.$$

□

Remark 4.7. The fact that in (3.8) we have kept $\beta_j(\xi)$ instead of $\beta_j(0)$ precludes similar higher-order estimates; indeed, it may be readily checked that application of D^q does *not* enhance decay by $(1+t)^{-q}$.

The previous linear estimates may be transposed to the nonlinear level as follows.

Corollary 4.8. *Assuming (H1)-(H3) and (D1)-(D3), the phase ψ of Theorem 1.10 and the mean M of Theorem 1.12 satisfy for all $t \geq 0$, $2 \leq p \leq \infty$,*

$$\|\psi_t(t) + \bar{k}dc_{|(\bar{M},\bar{k})}(M(t), \bar{k}\psi_x(t))\|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-3/4} \tag{4.21}$$

$$\|\partial_x^l \partial_t^m D\,(M(t), \bar{k}\psi_x(t))\|_{L^p(\mathbb{R})} \lesssim E_0 (1+t)^{-3/4}, \qquad 0 \leq l + 2m + 2 \leq K - 1\ .$$

Proof. Completely similar to the proofs of (3.38) in Proposition 3.8 and (4.16) in Proposition 4.5. □

Now the next lemma pulls out the dominant part of $\mathcal{N}(t)$.

Lemma 4.9. *Assuming (H1)–(H3), (D1)–(D3), we have*

$$\begin{aligned}\mathcal{N}(t) = \partial_x\ &\big(M(t)^T f_{MM}^{\mathrm{p}} \cdot M(t) + \bar{k}\psi_x(t) f_{kM}^{\mathrm{p}} \cdot M(t) + f_{kk}^{\mathrm{p}} (\bar{k}\psi_x(t))^2 + r_1(t)\big)\\ + \partial_t\ &\big(M(t)^T g_{MM}^{\mathrm{p}} \cdot M(t) + \bar{k}\psi_x(t) g_{kM}^{\mathrm{p}} \cdot M(t) + g_{kk}^{\mathrm{p}} (\bar{k}\psi_x(t))^2 + r_2(t)\big)\,,\end{aligned}$$

where f_{jk}^{p}, g_{jk}^{p} are periodic of period 1 *and $\|r_j(t)\|_{L^1(\mathbb{R})} \lesssim E_0^2 \ln(2+t)(1+t)^{-1}$, explicitly*

$$\begin{aligned}f_{MM}^{\mathrm{p}} &= -\bar{k}\,\frac{1}{2} d^2 f(\bar{U})(\partial_M \bar{U}, \partial_M \bar{U}) + \bar{k}\partial_M \bar{c}\ \partial_M \bar{U},\\ f_{kM}^{\mathrm{p}} &= -\bar{k}\, d^2 f(\bar{U})(\partial_k \bar{U}, \partial_M \bar{U}) + \bar{k}\partial_k \bar{c}\ \partial_M \bar{U} + \bar{k}\partial_k \bar{U}\ \partial_M \bar{c} + \bar{k}\partial_M \bar{U}',\\ f_{kk}^{\mathrm{p}} &= -\bar{k}\,\frac{1}{2} d^2 f(\bar{U})(\partial_k \bar{U}, \partial_k \bar{U}) + \bar{k}\partial_k \bar{c}\ \partial_k \bar{U} + \bar{U}' + \bar{k}\partial_k \bar{U}'\end{aligned}$$

and

$$g_{MM}^{\mathrm{p}} = 0, \qquad g_{kM}^{\mathrm{p}} = \frac{1}{\bar{k}} \partial_M \bar{U}, \qquad g_{kk}^{\mathrm{p}} = \frac{1}{\bar{k}} \partial_k \bar{U}.$$

Proof. This follows by a direct, but tedious, computation, using estimates (3.38), (4.15), (4.16) and (4.21), formulas from Lemma 2.3 and $v = z + \partial_k \bar{U}\, \bar{k}\psi_x + \partial_M \bar{U} \cdot M$. □

4.5. **Comparison with the linearized Whitham equations.** Now we begin comparisons with the Whitham equations by proving that the dynamics described by $s^{\mathrm{p}}(t)$ are well-approximated by the evolution of the linearized Whitham equations. To this end, we linearize about $w \equiv 0$ the quadratic approximant of the second-order Whitham system (1.7) and get

$$w_t + \bar{k}A_* w_x = \bar{k}^2 \tilde{B}_* w_{xx}. \tag{4.22}$$

We begin with some observations about (4.22).

Lemma 4.10 ([48, 49])**.** *Assuming (H1)–(H3), the coefficients a_j, b_j of the expansion $\lambda_j(\xi) = -i\bar{k}\xi a_j + (i\bar{k}\xi)^2 b_j + \mathcal{O}(\xi^3)$ given in* (3.3) *are given by the eigenvalues of the simultaneously-diagonalized coefficient matrices A_*, $\tilde{B}_*$, respectively, of the linearized quadratic approximant* (4.22)*.*

Proof. The first-order relation has already been established above in Proposition 1.7. The second-order relation essentially follows from the fact that the processes of linearization and formal expansion commute. The exact computations depend on whether we obtained the modulation system with the strategy followed in Section B.1.1 or with the one in Section B.1.2. In the latter case, computations are rather light while in the former case they are quite tedious but completely similar to those in [48, 49] in the context of the Saint-Venant and Korteweg-de Vries/Kuramoto-Sivashinsky equations. See also the proof given in [18] in the case of the Kuramoto-Sivashinsky equation. □

Remark 4.11. It is worth mentioning that the diffusion matrix of (1.7) is uniquely determined *only up to asymptotic equivalence*, i.e., it is $\tilde{B}_*$ and not B_* that is uniquely returned by the process of formal expansion. The relation between b_j and $\sigma(\tilde{B}_*)$ was used in [18] as a means to verify the spectral stability assumption (D2) for the critical eigenvalues $\Re\lambda_j(\xi) \sim \bar{k}^2\Re b_j\xi^2$. However, in the more recent studies [5, 6], we find it more convenient to instead verify (D2) entirely by numerical Evans function study, determining a_j b_j at the same time by numerical Taylor expansion.

Lemma 4.12. *Assuming (H1)–(H3), the solution operator $\Sigma(t)$ of the linearized quadratic approximant* (4.22) *is given by*

$$\Sigma(t) \; = \; \sum_{j=1}^{n+1} \sigma_j(t) V_j \tilde{V}_j^T,$$

where the $\sigma_j(t)$ are solution operators of the convected heat equations

$$u_t + a_j \bar{k} u_x = b_j \bar{k}^2 u_{xx},$$

and the vectors V_j and $\tilde{V}_j$ are defined in (4.17).

Proof. This follows by a straightforward diagonalization argument. □

We now come to linear comparisons, i.e. we aim to verify that the dynamics described by $s^p(t)$ are well-approximated by the evolution of the linearized Whitham equations (4.22). To write the comparison results as compactly as possible, we introduce the operator $\mathcal{I}$ defined by

$$\widehat{\mathcal{I}g}(\xi) \; = \; \frac{1}{i\xi}\widehat{g}(\xi). \tag{4.23}$$

Note that $\mathcal{I}$ can be identified as the convolution with a step function, and hence takes $L^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$.

Proposition 4.13. *Assuming (H1)–(H3) and (D1)–(D3), let $\Sigma(t)$ be the solution operator of* (4.22) *and g be a periodic function on $[0,1]$, $g \in H^1([0,1])$. Then, for all $t \geq 0$, $2 \leq p \leq \infty$,*

$$\begin{aligned} &\left\| \partial_x^l s^{\mathrm{p}}(t)(h_0 \bar{U}' + d) - \frac{1}{\bar{k}}\Sigma(t)\mathcal{I}\partial_x^l \begin{pmatrix} d - (\bar{U} - \langle \bar{U} \rangle)\partial_x h_0 \\ \bar{k}\partial_x h_0 \end{pmatrix} \right\|_{L^p(\mathbb{R})} \\ &\qquad \lesssim \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} t^{-\frac{l-1}{2}} (\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})} + \|d\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}), & l \geq 1 \\ (1+t)^{-\frac{1}{2}(1-1/p)} (\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} + \|d\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}), & l = 0 \end{cases}; \end{aligned} \tag{4.24}$$

$$\begin{aligned} &\left\| \partial_x^l s^{\mathrm{p}}(t)\partial_x(h_0 g) - \frac{1}{\bar{k}}\Sigma(t)\mathcal{I}\partial_x^l \begin{pmatrix} [\langle g \rangle - i\sum_{l=1}^{n} \langle \partial_\xi \tilde{q}_l(0), \partial_x g \rangle e_l]\partial_x h_0 \\ \bar{k}\langle \tilde{q}_{n+1}(0), \partial_x g \rangle \partial_x h_0 \end{pmatrix} \right\|_{L^p(\mathbb{R})} \\ &\qquad \lesssim \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} t^{-\frac{l-1}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}, & l \geq 1 \\ (1+t)^{-\frac{1}{2}(1-1/p)} \|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}, & l = 0 \end{cases}; \end{aligned} \tag{4.25}$$

$$\begin{aligned} &\left\| \partial_x^{l+1} s^{\mathrm{p}}(t)(h_0 g) - \frac{1}{\bar{k}}\Sigma(t)\mathcal{I}\partial_x^l \begin{pmatrix} \langle g \rangle \partial_x h_0 \\ 0 \end{pmatrix} \right\|_{L^p(\mathbb{R})} \\ &\qquad \lesssim \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} t^{-\frac{l-1}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}, & l \geq 1 \\ (1+t)^{-\frac{1}{2}(1-1/p)} \|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}, & l = 0 \end{cases}. \end{aligned} \tag{4.26}$$

Proof. (i) (Proof of (4.24)*, case $h_0 = 0$).* Following the proof of (3.14) in Proposition 3.3, we obtain that the difference between $\partial_x^l s^{\rm p}(t)(d)$ and the function

$$x \longmapsto \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-i\bar{k}a_j\xi - \bar{k}^2 b_j \xi^2)t} \frac{1}{i\bar{k}\xi} (i\xi)^l \beta^{(j)}(0) \langle \tilde{\phi}_j(0,\cdot), \check{d}(\xi,\cdot)\rangle_{L^2([0,1])} d\xi \tag{4.27}$$

is bounded in $L^p(\mathbb{R})$ by $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l}{2}} \|d\|_{L^1(\mathbb{R})}$. Since for $1 \le j \le n+1$ and $|\xi| \le \pi$ we have

$$\beta^{(j)}(0)\langle \tilde{\phi}_j(0,\cdot), \check{d}(\xi,\cdot)\rangle_{L^2([0,1])} \;=\; V_j \tilde{V}_j \cdot \begin{pmatrix} \langle \check{d}(\xi,\cdot)\rangle \\ 0 \end{pmatrix} \;=\; V_j \tilde{V}_j \cdot \begin{pmatrix} \widehat{d}(\xi) \\ 0 \end{pmatrix},$$

the function in (4.27) is recognized to be the low-frequency part of $\frac{1}{k}\Sigma(t)\mathcal{I}\partial_x^l \begin{pmatrix} d \\ 0 \end{pmatrix}$, whose high-frequency part is still to be bounded. When $l = 0$, we bound it in $L^p(\mathbb{R})$ by $Ce^{-\eta t}\|d\|_{L^{1/(1/p+1/2)}(\mathbb{R})}$ using, for $1 \le j \le n+1$

$$\left\| \xi \mapsto e^{-\bar{k}^2 b_j \xi^2 t} \xi^{-1} \right\|_{L^2(\mathbb{R}\setminus[-\pi,\pi])} \lesssim e^{-\eta t}$$

for some $\eta > 0$. When $l \ge 1$, however, we bound the high-frequency part of $\frac{1}{k}\Sigma(t)\mathcal{I}\partial_x^l \begin{pmatrix} d \\ 0 \end{pmatrix}$ in $L^p(\mathbb{R})$ by $Ct^{-\frac{l-1}{2}} e^{-\eta t}\|d\|_{L^p(\mathbb{R})}$, recognizing it as the convolution of d with a kernel that is bounded pointwise (using Haussdorff-Young estimates) by

$$x \;\longmapsto\; C\, t^{-\frac{l}{2}} e^{-\eta\, t} \frac{1}{1+\frac{x^2}{t}},$$

which is bounded in $L^1(\mathbb{R})$ by $Ct^{-\frac{l-1}{2}} e^{-\eta\, t}$, for some $\eta > 0$.

(ii) (Proof of (4.24)*, case $d = 0$).* Following the proof of (3.23) in Proposition 3.5, we obtain that the difference between $\partial_x^l s^{\rm p}(t)(h_0\bar{U}')$ and

$$\begin{aligned} x \longmapsto & \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-i\bar{k}a_j\xi - \bar{k}^2 b_j\xi^2)t} \frac{1}{i\bar{k}\xi} (i\xi)^l \beta^{(j)}(0) \langle \tilde{\phi}_j(0,\cdot), \bar{U}'[\check{h}_0(\xi,\cdot) - \widehat{h}_0(\xi)]\rangle_{L^2([0,1])} d\xi \\ & + \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-i\bar{k}a_j\xi - \bar{k}^2 b_j\xi^2)t} \frac{1}{i\bar{k}\xi} (i\xi)^l \beta^{(j)}(0) \langle \tilde{\phi}_j(0,\cdot) + \xi\, \partial_\xi \tilde{\phi}_j(0,\cdot), \bar{U}'\widehat{h}_0(\xi)\rangle_{L^2([0,1])} d\xi \end{aligned} \tag{4.28}$$

is bounded in $L^p(\mathbb{R})$ by $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})}$. Now, as in the proofs of (3.24)-(3.25) in Proposition 3.5, we observe that, for $1 \le j \le n+1$ and $\xi \in [-\pi, \pi]$,

$$\langle \tilde{\phi}_j(0), \bar{U}'[\check{h}_0(\xi) - \widehat{h}_0(\xi)]\rangle = -\tilde{V}_j \cdot \begin{pmatrix} [(\bar{U} - \langle \bar{U}\rangle)\partial_x h_0]^\wedge(\xi) \\ 0 \end{pmatrix} + i\xi\, \tilde{V}_j \cdot \begin{pmatrix} \langle \bar{U}, [\check{h}_0(\xi) - \widehat{h_0}(\xi)]\rangle \\ 0 \end{pmatrix}$$

$$\langle \tilde{\phi}_j(0) + \xi\, \partial_\xi \tilde{\phi}_j(0), \bar{U}'\widehat{h}_0(\xi)]\rangle = \xi \langle \partial_\xi \tilde{\phi}_j(0), \bar{U}'\rangle\; \widehat{h}_0(\xi) \;=\; \tilde{V}_j \cdot \begin{pmatrix} 0 \\ \bar{k}\partial_x h_0 \end{pmatrix}.$$

Up to a term that is also bounded in $L^p(\mathbb{R})$ by $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})}$, the function in (4.28) is recognized as the low-frequency part of $\frac{1}{k}\Sigma(t)\mathcal{I}\partial_x^l \begin{pmatrix} -(\bar{U} - \langle U\rangle)\partial_x h_0 \\ \bar{k}\partial_x h_0 \end{pmatrix}$. The remaining high-frequency part is bounded as in the case $h_0 = 0$ above. This completes the proof of (4.24) in the case $d = 0$ and by linearity the proof of (4.24) in any case.

(iii) (Proof of (4.25)*).* Combining elements of the proofs of (3.14) in Proposition 3.3 and (3.23) in Proposition 3.5, we obtain that the difference between $\partial_x^l s^{\mathrm{p}}(t)\partial_x(h_0 g)$ and

$$x \longmapsto \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-i\bar{k}a_j\xi - \bar{k}^2 b_j \xi^2)t} \frac{1}{i\bar{k}\xi}(i\xi)^l \beta^{(j)}(0)\langle\tilde{\phi}_j(0,\cdot), i\xi\widehat{h}_0(\xi)g\rangle_{L^2([0,1])} d\xi$$
$$+ \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-i\bar{k}a_j\xi - \bar{k}^2 b_j \xi^2)t} \frac{1}{i\bar{k}\xi}(i\xi)^l \beta^{(j)}(0)\langle -\xi\, \partial_x\partial_\xi\tilde{\phi}_j(0,\cdot), \widehat{h}_0(\xi)g\rangle_{L^2([0,1])} d\xi$$

is bounded in $L^p(\mathbb{R})$ by $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})}$. Since, for $1 \leq j \leq n+1$ and $\xi \in [-\pi,\pi]$,

$$\begin{aligned} \langle\tilde{\phi}_j(0,\cdot), i\xi\widehat{h}_0(\xi)g\rangle &= \tilde{V}_j \cdot \begin{pmatrix} \langle g\rangle\widehat{\partial_x h_0}(\xi) \\ 0 \end{pmatrix} \\ -\xi\langle\partial_x\partial_\xi\tilde{\phi}_j(0,\cdot), \widehat{h}_0(\xi)g\rangle &= \tilde{V}_j \cdot \begin{pmatrix} -i\displaystyle\sum_{l=1}^{n}\langle\partial_\xi\tilde{q}_l(0), \partial_x g\rangle e_l \\ \bar{k}\langle\tilde{q}_{n+1}(0), \partial_x g\rangle \end{pmatrix} \widehat{\partial_x h_0}(\xi), \end{aligned} \tag{4.29}$$

the function in (4.28) is the low-frequency part of

$$\frac{1}{\bar{k}}\Sigma(t)\mathcal{I}\partial_x^l \begin{pmatrix} [\langle g\rangle - i\displaystyle\sum_{l=1}^{n}\langle\partial_\xi\tilde{q}_l(0), \partial_x g\rangle e_l]\partial_x h_0 \\ \bar{k}\langle\tilde{q}_{n+1}(0), \partial_x g\rangle\partial_x h_0 \end{pmatrix}$$

whose high-frequency part is bounded as above.

(iv) (Proof of (4.26)*).* Following the proof of (3.23) in Proposition 3.5, we obtain that the difference between $\partial_x^{l+1} s^{\mathrm{p}}(t)(h_0 g)$ and

$$x \longmapsto \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-i\bar{k}a_j\xi - \bar{k}^2 b_j \xi^2)t} \frac{1}{i\bar{k}\xi}(i\xi)^{l+1} \beta^{(j)}(0)\langle\tilde{\phi}_j(0,\cdot), \widehat{h_0}(\xi)g\rangle_{L^2([0,1])} d\xi$$

is bounded in $L^p(\mathbb{R})$ by $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})}$. Again this function is the low-frequency part of the expected term, whose high-frequency part may be bounded as in the proof of (4.24) in the case $h_0 = 0$. $\square$

Note that, since A_* and $\Sigma(t)$ commute, combining (4.26) with (4.19) we obtain for any periodic function g the bound

$$\begin{aligned} &\|\partial_x^l \partial_t s^{\mathrm{p}}(t)(h_0 g) + \Sigma(t) A_* \partial_x^l \begin{pmatrix} \langle g\rangle h_0 \\ 0 \end{pmatrix}\|_{L^p(\mathbb{R})} \\ &\qquad \lesssim \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} t^{-\frac{l-1}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}, & l \geq 1, \\ (1+t)^{-\frac{1}{2}(1-1/p)}\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}, & l = 0. \end{cases} \end{aligned}$$

4.6. **Nonlinear connection to the Whitham equations.** Before proving that the local means, wavenumber, and phase are indeed well-approximated by solutions of the appropriate Whitham equations, it is necessary to compute some averages involving the main part of the nonlinear term $\mathcal{N}$ identified in Lemma 4.9. This is the purpose of the next lemma.

Lemma 4.14. *Assuming (H1)–(H3) and (D1)–(D3), we have the identities*

$$\frac{1}{\bar{k}}\begin{pmatrix}\langle f^{\rm p}_{MM}\rangle - i\displaystyle\sum_{l=1}^{n}\langle\partial_\xi\tilde{q}_l(0),\partial_x f^{\rm p}_{MM}\rangle e_l\\ \bar{k}\langle\tilde{q}_{n+1}(0),\partial_x f^{\rm p}_{MM}\rangle\end{pmatrix} = \begin{pmatrix}-\frac12\partial_M^2 F_{|(\bar M,\bar k)}+\partial_M c_{|(\bar M,\bar k)}\\ \frac12\partial_M^2\omega_{|(\bar M,\bar k)}\end{pmatrix},$$

$$\frac{1}{\bar{k}}\begin{pmatrix}\langle f^{\rm p}_{kM}\rangle - i\displaystyle\sum_{l=1}^{n}\langle\partial_\xi\tilde{q}_l(0),\partial_x f^{\rm p}_{kM}\rangle e_l\\ \bar{k}\langle\tilde{q}_{n+1}(0),\partial_x f^{\rm p}_{kM}\rangle\end{pmatrix} - A_*\begin{pmatrix}\langle g^{\rm p}_{kM}\rangle\\ 0\end{pmatrix} =$$

$$\begin{pmatrix}-\partial_{kM}^2 F_{|(\bar M,\bar k)}+\partial_k c_{|(\bar M,\bar k)}{\rm Id}-\frac{1}{\bar k}\left(\partial_M F_{|(\bar M,\bar k)}-\bar c{\rm Id}\right)\\ \partial_{kM}^2\omega_{|(\bar M,\bar k)}\end{pmatrix},$$

$$\frac{1}{\bar{k}}\begin{pmatrix}\langle f^{\rm p}_{kk}\rangle - i\displaystyle\sum_{l=1}^{n}\langle\partial_\xi\tilde{q}_l(0),\partial_x f^{\rm p}_{kk}\rangle e_l\\ \bar{k}\langle\tilde{q}_{n+1}(0),\partial_x f^{\rm p}_{kk}\rangle\end{pmatrix} - A_*\begin{pmatrix}\langle g^{\rm p}_{kk}\rangle\\ 0\end{pmatrix} = \begin{pmatrix}-\frac12\partial_k^2 F_{|(\bar M,\bar k)}-\frac{1}{\bar k}\partial_k F_{|(\bar M,\bar k)}\\ \frac12\partial_k^2\omega_{|(\bar M,\bar k)}\end{pmatrix}.$$

Proof. Simple means are computed directly, for example

$$A_*\begin{pmatrix}\langle g^{\rm p}_{kM}\rangle\\ 0\end{pmatrix} = \frac{1}{\bar k}\begin{pmatrix}\partial_M F_{|(\bar M,\bar k)}-\bar c{\rm Id}\\ \bar k\partial_M c_{|(\bar M,\bar k)}\end{pmatrix}.$$

To compute the remaining terms, we first use the identity ${\rm Id}=\sum_{j=1}^{n+1}V_j\tilde V_j^T$ and come back to (4.29). Using the algebraic identities (A.3)(iii), (A.4), (A.5) and (A.6), then, we find that

$$\begin{aligned}\partial_x f^{\rm p}_{MM} &= \frac12\bar U'\ \partial_M^2\omega_{|(\bar M,\bar k)}\ -\ \frac12 L_0\partial_M^2 U_{|(\bar M,\bar k)},\\ \partial_x f^{\rm p}_{kM} &= -\bar U'\ \bar k\partial_{kM}^2 c_{|(\bar M,\bar k)}\ -\ L_0\partial_{kM}^2 U_{|(\bar M,\bar k)},\\ \partial_x f^{\rm p}_{kk} &= \frac12\bar U'\ \partial_k^2\omega_{|(\bar M,\bar k)}\ -\ \frac12 L_0\partial_M^2 U_{|(\bar M,\bar k)}\ -\ \frac{1}{\bar k}L_0\partial_k U_{|(\bar M,\bar k)}.\end{aligned}$$

Now, observe that, for $1\le j\le n+1$, expanding the fact that for, $|\xi|\le\xi_0$, $\tilde\phi_j(\xi)$ is a left eigenfunction of L_ξ associated to $\lambda_j(\xi)$ yields for any periodic g the identity

$$i\langle\partial_\xi\tilde\phi_j(0),L_0 g\rangle\ =\ \bar k\langle\tilde\phi_j(0),L^{(1)}g\rangle+a_j\bar k\langle\tilde\phi_j(0),g\rangle\ =\ \bar k\tilde V_j\cdot\begin{pmatrix}\langle L^{(1)}g\rangle\\ 0\end{pmatrix}+\bar k\tilde V_j\cdot A_*\begin{pmatrix}\langle g\rangle\\ 0\end{pmatrix}.$$

Since, for $1\le j\le n+1$,

$$i\langle\partial_\xi\tilde\phi_j(0),\bar U'\rangle\ =\ -\bar k,$$

the proof of the lemma is then achieved by simple direct computations. □

We have now in hand all the pieces needed to achieve the proof of Theorem 1.12.

Proof of Theorem 1.12. Estimates (1.13) and (1.19) already follow from Proposition 4.5 together with Lemma 2.7.

Using Duhamel's principle we may write (B.36) as

$$\begin{pmatrix}M_W\\ k_W\end{pmatrix}(t)=\Sigma(t)\begin{pmatrix}d_0+m(h_0)\\ \bar k\partial_x h_0\end{pmatrix}+\int_0^t\Sigma(t-s)\partial_x\Big(\frac12\begin{pmatrix}M_W\\ k_W\end{pmatrix}^T\Gamma_*\begin{pmatrix}M_W\\ k_W\end{pmatrix}\Big)(s)ds\ ,$$

where $m(h_0):=-(\bar U-\bar M)\partial_x h_0$ and Σ is the constant-coefficient solution operator defined in Lemma 4.12. On the other hand, using Propositions 3.3, 3.5 and 4.13, and Lemmas 4.9, 4.12

and 4.14, we find that

$$\begin{pmatrix} M(t) \\ \bar k\psi_x(t)\end{pmatrix} = \Sigma(t)\begin{pmatrix} d_0+m(h_0) \\ \bar k\partial_x h_0\end{pmatrix} + \int_0^t \Sigma(t-s)\partial_x\Big(\frac{1}{2}\begin{pmatrix} M \\ \bar k\psi_x\end{pmatrix}^T \Gamma_* \begin{pmatrix} M \\ \bar k\psi_x\end{pmatrix}\Big)(s)ds + \tilde r^{\mathrm{p}}(t),$$

where the residual $r^{\mathrm{p}}(t)$ satisfies the bound

$$\|\tilde r^{\mathrm{p}}(t)\|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-\frac12(1-1/p)-\frac12}, \quad 2\le p\le\infty,$$

where E_0 is defined as in Theorem 1.10. Thus, subtracting and defining $\delta := (M, \bar k\psi_x) - (M_W, k_W)$, we have

$$\delta(t) = \int_0^t \Sigma(t-s)\partial_x\Big(\frac12 \delta^T\Gamma_* \begin{pmatrix} M \\ \bar k\psi_x\end{pmatrix} + \frac12\begin{pmatrix} M_W \\ k_W\end{pmatrix}^T \Gamma_*\delta\Big)(s)ds + \tilde r^{\mathrm{p}}(t).$$

Letting $\eta>0$ be fixed but arbitrary and defining

$$\nu(t) := \sup_{p\in[2,\infty]}\ \sup_{0\le s\le t} \|\delta(s)\|_{L^p(\mathbb{R})}(1+s)^{\frac12(1-1/p)+\frac12-\eta},$$

we thus obtain by the standard bounds (see [41]) $\|\Sigma(t)\partial_x f\|_{L^p(\mathbb{R})} \le Ct^{-\frac12(1/q-1/p)-\frac12}\|f\|_{L^q(\mathbb{R})}$ when $1\le q\le p\le\infty$, that, for any $2\le p\le\infty$ and all $t>0$,

$$\begin{aligned}\|\delta(t)\|_{L^p(\mathbb{R})} &\lesssim E_0\ln(2+t)\ (1+t)^{-\frac12(1-1/p)-\frac12} + \int_0^{t/2}(t-s)^{-\frac12(1-1/p)-\frac12}\nu(t)E_0(1+s)^{-1+\eta}ds \\ &\quad + \int_{t/2}^t (t-s)^{-\frac12(1/2-1/p)-\frac12}\nu(t)E_0(1+s)^{-\frac54+\eta}ds \\ &\lesssim E_0(\nu(t)+1)(1+t)^{-\frac12(1-1/p)-\frac12+\eta},\end{aligned}$$

which in turn yields $\nu(t)\le C_\eta E_0\left(1+\nu(t)\right)$. Thus, if $E_0 < 1/(2C_\eta)$ it follows that $\nu(t)\le 2C_\eta E_0$ for all $t\ge 0$. This provides the needed bounds on $(M-M_W, \bar k\psi_x - k_W)$. Using these bounds a simpler computation yields the result for $\psi_W-\psi$. □

Remark 4.15. Though the computation of quadratic coupling coefficients is heavy going, we note that already from Proposition 4.13 and Lemma 4.12 one may conclude that long-time behavior is governed to leading order by *some* "Whitham-like" system (B.27), with no computation at all, since in the Duhamel formulation the principal part of nonlinear terms factors on the left as $\int_0^t \Sigma(t-s)\partial_x(\dots)(s)ds$.

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Appendix A. Algebraic relations

We record in this appendix some crucial relations obtained by differentiating the profile equations. In order to differentiate, we here consider variable parameters (M,k) rather than fixed values $(\bar M, \bar k)$, imposing dependence implicitly through the profile equations (denoting $\langle a\rangle := \int_0^1 a$):

$$k^2U'' - k(f(U))' + kcU' = 0, \qquad \langle U\rangle = M. \tag{A.1}$$

We expand $L_\xi = L_0 + ik\xi L^{(1)} + (ik\xi)^2 L^{(2)}$ with

$$\text{(A.2)} \qquad \begin{array}{rcl} L_0 v &=& k^2 v'' - k((df)(U)\, v)' + kc\, v' \ ; \\ L^{(1)} v &=& 2kv' - (df)(U)\, v + c\, v \ ; \\ L^{(2)} v &=& v \ . \end{array}$$

Then, by differentiation of (A.1), we obtain

$$\text{(A.3)} \qquad \begin{array}{lrcl} L_0\, U' = 0 \ , & \langle U' \rangle &=& 0 \ ; \\ L_0\, \partial_M U + kU'\, \partial_M c = 0 \ , & \langle \partial_M U \rangle &=& \mathrm{Id} \ ; \\ L_0\, \partial_k U + kU'\, \partial_k c + L^{(1)} U' = 0 \ , & \langle \partial_k U \rangle &=& 0 \ . \end{array}$$

Accordingly, with $\omega = -kc$, using $L^{(1)} U' = kU''$, we have

$$< d^2 U >= 0 \ ,$$

$$\text{(A.4)} \qquad L_0\, \partial_M^2 U + 2k\, (\partial_M c)\, (\partial_M U)' - k\, [(d^2 f)(U)(\partial_M U, \partial_M U)]' = -U'\, k\, \partial_M^2 c \ ,$$

$$\text{(A.5)} \qquad \begin{array}{rl} L_0\, \partial_{kM}^2 U & +\ \ k\, (\partial_k c)\, (\partial_M U)' + k\, (\partial_M c)\, (\partial_k U)' - k\, [(d^2 f)(U)(\partial_k U, \partial_M U)]' \\ & +\ \ k\, (\partial_M U)'' = -U'\, k\, \partial_{kM}^2 c \ , \end{array}$$

$$\text{(A.6)} \qquad L_0\, \partial_k^2 U + 2k\, (\partial_k c)\, (\partial_k U)' - k\, [(d^2 f)(U)(\partial_k U, \partial_k U)]' + 2k\, (\partial_k U)'' = -U'\, k\, \partial_k^2 c \ .$$

Appendix B. The Whitham equations and asymptotic equivalence

In this appendix, we explain how to obtain the needed formal averaged modulation system for comparison to our analytical description of asymptotic behavior. This is performed in three steps.

(1) First, we develop a direct WKB-like formal approximation. At this stage we obtain a system that may contain harmless irrelevant terms.
(2) Next, we use known results about large-time asymptotic behavior of systems of conservation laws about constant states to get a canonical form for the averaged modulation system.
(3) Finally, we adapt the system taking into account the fact that the analysis of the main part of the paper is carried out after an implicit nonlinear change of coordinates.

B.1. **Formal asymptotics.** Though the full nonlinear analysis may be carried out without distinction between linearly coupled and linearly uncoupled cases, the formal derivation of averaged equations involves resolutions of systems of the form $L_0 g = h$ and therefore requires knowledge of the kernel of L_0. We are thus compelled to provide two separate derivations.

Besides, there are at least two ways to obtain relevant averaged equations. The first one is to develop a full WKB-type expansion as in [48, 49], extending the procedure in [62] to get higher order equations. This method provides the hyperbolic part of the averaged system in a quick way and a nice form. Its main drawback is that it requires a knowledge of the kernel of L_0 for all waves close to the wave under study essentially reducing the scope of the method to the nondegenerate case or to a fully degenerate case where $\partial_M c$ would vanish in a neighborhood of the studied wave.[29] The second method is designed to study dynamics about a given wave, so that it does not suffer from the same flaws; moreover, it is closer to our nonlinear analysis, and yields a semilinear system.

We derive the system for the generic case with the first method and the one for the linearly uncoupled case with the second one. Note that both methods provide averaged systems with diffusion matrices containing terms that are not relevant for our present analysis.

[29]A situation that trivially occurs when some symmetry is present, see Remark 1.27.

B.1.1. *Generic case.* To treat the linearly phase-coupled case, we essentially borrow the derivation of [49] for the Korteweg-de Vries/Kuramoto-Sivashinsky equation, a model for which linear phase-coupling is a consequence of assumptions (H1)-(H2) and (D3). In the present derivation, we assume that all the waves involved in the slow-modulation description satisfy (H1)-(H2) and (D3) and are linearly phase-coupled.

Since in this derivation there is no reference wave, thus no privileged frame, we go back to the original equation

$$u_t \ + \ f(u)_x \ = \ u_{xx}. \tag{B.1}$$

We are looking for a formal expansion of a solution u of equation (B.1) according to the two-scale *ansatz*

$$u(x,t) \ = \ U\left(\frac{\Psi(\varepsilon x, \varepsilon t)}{\varepsilon}; \varepsilon x, \varepsilon t\right) \tag{B.2}$$

where

$$U(y,X,T) = \sum_j \varepsilon^j U_j(y;X,T) \quad \text{and} \quad \Psi(X,T) = \sum_j \varepsilon^j \Psi_j(X,T) \ , \tag{B.3}$$

with the functions U and U_j being 1-periodic in the y-variable. We insert the *ansatz* (B.2,B.3) into (B.1) and collect terms of the same order in ε.

First this yields, with $\Omega_0 = \partial_T \phi_0$ and $\kappa_0 = \partial_X \Psi_0$, $\Omega_0 \, \partial_y U_0 + \kappa_0 \, \partial_y(f(U_0)) = \kappa_0^2 \, \partial_y^2 U_0$, which is solved by

$$\begin{aligned} \Omega_0(X,T) &= -k_0(X,T) \ c(\mathcal{M}_0(X,T), \kappa_0(X,T)), \\ U_0(y;X,T) &= \quad U(y; \mathcal{M}_0(X,T), \kappa_0(X,T)). \end{aligned} \tag{B.4}$$

We have disregarded in (B.4) the possibility of a phase shift dependent on (X,T) since this is already encoded by Ψ_1. We will have to rule out similar problems of uniqueness in the following steps. At this stage the compatibility condition $\partial_T \partial_X \Psi_0 = \partial_X \partial_T \Psi_0$ already yields the first equation of a Whitham's modulation system:

$$\partial_T \kappa_0 + \partial_X \left(\kappa_0 \, c(\mathcal{M}_0, \kappa_0)\right) \ = \ 0 \ . \tag{B.5}$$

In the rest of the derivation, we will use the notations of Proposition 3.1 and Appendix A, with the convention that operators act in y and are associated to the wave profile $U(\,\cdot\,; \mathcal{M}_0(X,T), \kappa_0(X,T))$. To fix some of the uniqueness issues of the *ansatz*, we pick, for any (M,k), $u^{adj}(\,\cdot\,; M,k)$ a generalized zero eigenfunction of L_0^* such that $\langle u^{adj}, \partial_M U_{|(M,k)} \rangle = 0$ and $\langle u^{adj}, U'(\,\cdot\,; M,k) \rangle = 1$, set $u_0^{adj}(y;X,T) = u^{adj}(\,\cdot\,; \mathcal{M}_0(X;T), \kappa_0(X,T))$ and add to the *ansatz* the normalizing condition

$$\langle u_0^{adj}, U_j(\,\cdot\,; X,T) \rangle \ = \ 0, \qquad j \neq 0. \tag{B.6}$$

The next step of the identification process gives, with $\Omega_1 = \partial_T \Psi_1$ and $\kappa_1 = \partial_X \Psi_1$,

$$\begin{aligned} (\Omega_1 + c(\mathcal{M}_0, \kappa_0)\kappa_1)\partial_y U_0 \ &- \ \kappa_1 \, L^{(1)} \partial_y U_0 \ - \ L_0 U_1 \ - \ L^{(1)} \partial_X U_0 \\ &- \ \partial_X \kappa_0 \, L^{(2)} \partial_y U_0 \ + \ \partial_T U_0 + c(\mathcal{M}_0, \kappa_0) \partial_X U_0 \ = \ 0, \end{aligned} \tag{B.7}$$

whose solvability condition reads

$$\partial_T \mathcal{M}_0 + \partial_X (F(\mathcal{M}_0, \kappa_0)) \ = \ 0, \tag{B.8}$$

where F denotes the averaged flux $F(M,k) = \langle f(U(\,\cdot\,;M,k))\rangle$. To proceed, for arbitrary (M,k) we introduce $g^k(\,\cdot\,;M,k)$, $g^M(\,\cdot\,;M,k)$ solutions of

$$\text{(B.9)}\quad \begin{aligned} L_0(g^k(\,\cdot\,;M,k)) \;=\; & -\, L^{(1)}\partial_k U_{|(M,k)} - \partial_k F_{|(M,k)} \;-\; L^{(2)} U'_{|(M,k)} \\ & -\, \partial_k U_{|(M,k)}\; k\,\partial_k c_{|(M,k)} \;-\; (\partial_M U_{|(M,k)} - \mathrm{Id})\;\partial_k F_{|(M,k)} \end{aligned}$$

$$\text{(B.10)}\quad \begin{aligned} L_0(g^M(\,\cdot\,;M,k)) \;=\; & -L^{(1)}\partial_M U_{|(M,k)} - \partial_M F_{|(M,k)} + c_{|(M,k)}\mathrm{Id} \\ & -\partial_k U_{|(M,k)}\; k\,\partial_M c_{|(M,k)} - (\partial_M U_{|(M,k)} - \mathrm{Id})[\partial_M F_{|(M,k)} - c_{|(M,k)}\mathrm{Id}] \end{aligned}$$

orthogonal to $u^{adj}(\,\cdot\,;M,k)$ and set $g = \begin{pmatrix} g^M & g^k \end{pmatrix}$ and $g_0(\,\cdot\,;X,T) = g(\,\cdot\,;\mathcal{M}_0(X,T),\kappa_0(X,T))$. Then with (B.5)-(B.8) and (A.3) equation (B.7) reads

$$\text{(B.11)}\qquad \begin{aligned} L_0\left(U_1 \;-\; dU_{|(\mathcal{M}_0,\kappa_0)}(\,\cdot\,;\tilde{\mathcal{M}}_1,\kappa_1) - g_0\,\begin{pmatrix}\partial_X\mathcal{M}_0\\ \partial_X\kappa_0\end{pmatrix}\right) \\ = \;(\Omega_1 \;+\; \kappa_0 dc_{|(\mathcal{M}_0,\kappa_0)}(\tilde{\mathcal{M}}_1,\kappa_1) + c_{|(\mathcal{M}_0,\kappa_0)}\,\kappa_1)\,\partial_y U_0 \end{aligned}$$

for any choice of $\tilde{\mathcal{M}}_1$. Let us set $\mathcal{M}_1 = \langle U_1\rangle$. Choosing $\tilde{\mathcal{M}}_1$ to get

$$\text{(B.12)}\qquad \Omega_1 + \kappa_0\; dc(\mathcal{M}_0,\kappa_0)\,[\tilde{\mathcal{M}}_1,\kappa_1] + c(\mathcal{M}_0,\kappa_0)\,\kappa_1 = \;0$$

and normalizing the parametrization, as in Lemma 4.1, to get, for any (M,k),

$$\text{(B.13)}\qquad \langle u^{adj}(\,\cdot\,;M,k), \partial_k U_{|(M,k)}\rangle = 0,$$

equation (B.11) is reduced to

$$\begin{aligned} U_1 \;&=\; dU_{|(\mathcal{M}_0,\kappa_0)}(\tilde{\mathcal{M}}_1,\kappa_1) \;+\; g_0\,\begin{pmatrix}\partial_X\mathcal{M}_0\\ \partial_X\kappa_0\end{pmatrix} \\ \mathcal{M}_1 \;&=\; \tilde{\mathcal{M}}_1 \;+\; \langle g_0\rangle\,\begin{pmatrix}\partial_X\mathcal{M}_0\\ \partial_X\kappa_0\end{pmatrix}. \end{aligned}$$

Then, compatibility condition $\partial_T\kappa_1 = \partial_X\Omega_1$ yields

$$\text{(B.14)}\qquad \begin{aligned} \partial_T\kappa_1 + \;\partial_X(\kappa_0 dc_{|(\mathcal{M}_0,\kappa_0)}[\mathcal{M}_1,\kappa_1] + c_{|(\mathcal{M}_0,\kappa_0)}\,\kappa_1) \;= \\ \partial_X\left(\kappa_0\partial_M c_{|(\mathcal{M}_0,\kappa_0)}\,\langle g_0\rangle\,\begin{pmatrix}\partial_X\mathcal{M}_0\\ \partial_X\kappa_0\end{pmatrix}\right). \end{aligned}$$

Returning to the identification process, we obtain an equation of the form

$$\partial_T U_1 \;+\; \partial_X(df(U_0)\,U_1) \;-\; \partial_X^2 U_0 \;-\; L_0 U_2 \;+\; \partial_y(\;\cdots\;) \;=\; 0,$$

whose solvability condition is

$$\text{(B.15)}\qquad \begin{aligned} \partial_T\mathcal{M}_1 \;+\; \partial_X(dF_{|(\mathcal{M}_0,\kappa_0)}[\mathcal{M}_1,\kappa_1]) \;=\; & \partial_X^2\mathcal{M}_0 \;-\; \partial_X\left(\langle df(U_0)\,g_0\rangle\,\begin{pmatrix}\partial_X\mathcal{M}_0\\ \partial_X\kappa_0\end{pmatrix}\right) \\ & +\,\partial_X\left(\partial_M F_{|(\mathcal{M}_0,\kappa_0)}\,\langle g_0\rangle\,\begin{pmatrix}\partial_X\mathcal{M}_0\\ \partial_X\kappa_0\end{pmatrix}\right). \end{aligned}$$

To write the second order system in a compact form, let us introduce, for arbitrary (M,k),

$$\begin{aligned} d_{1,1}(M,k) \;&=\; \mathrm{Id} - \langle df(U(M,k))\,g^M(M,k)\rangle + \partial_M F(M,k)\,\langle g^M(M,k)\rangle \\ d_{1,2}(M,k) \;&=\; -\langle df(U(M,k))\,g^k(M,k)\rangle + \partial_M F(M,k)\,\langle g^k(M,k)\rangle \\ d_{2,1}(M,k) \;&=\; k\partial_M c\,(M,k)\;\langle g^M(M,k)\rangle \\ d_{2,2}(M,k) \;&=\; k\partial_M c\,(M,k)\;\langle g^k(M,k)\rangle. \end{aligned} \qquad .$$

With these notations, systems (B.5,B.8), (B.14,B.15) coincide with the first systems obtained in the formal expansion of a solution $(\mathcal{M},\kappa)$ of

$$\text{(B.16)}\qquad \left\{\begin{array}{rcrcl} \partial_t\mathcal{M} & + & \partial_x(F(\mathcal{M},\kappa)) & = & \partial_x\left(d_{1,1}(\mathcal{M},\kappa)\,\partial_x\mathcal{M}+d_{1,2}(\mathcal{M},\kappa)\,\partial_x\kappa\right)\\ \partial_t\kappa & + & \partial_x(\kappa\, c(\mathcal{M},\kappa)) & = & \partial_x\left(d_{2,1}(\mathcal{M},\kappa)\,\partial_x\mathcal{M}+d_{2,2}(\mathcal{M},\kappa)\,\partial_x\kappa\right)\end{array}\right.$$

according to the slow *ansatz*

$$\text{(B.17)}\qquad (\mathcal{M},\kappa)(x,t)\;=\;\sum_j\varepsilon^j(\mathcal{M}_j,\kappa_j)(\varepsilon x,\varepsilon t).$$

We call system (B.16) a *(second-order) Whitham's modulation system.*

B.1.2. *Phase-decoupled case.* For the phase-decoupled case, we propose an alternative derivation that would also work for the uncoupled case. We pick a wave of parameters $(\bar M,\bar k)$ and assume that it satisfies (H1)-(H2) and (D3) and is linearly phase-decoupled.

We again insert the *ansatz* (B.2,B.3) into (B.1) and collect terms of the same order in ε but this time we specialize to $(\mathcal{M}_0,\kappa_0)=(\bar M,\bar k)$. We keep (B.6) as *ansatz* normalization and (B.13) as parametrization normalization. The first nontrivial equation is with $(\Omega_1,\kappa_1)=(\partial_T\Psi_1,\partial_X\Psi_1)$

$$\text{(B.18)}\qquad (\Omega_1+c(\bar M,\bar k)\kappa_1)\bar U'-\kappa_1\,L^{(1)}\bar U'\;-\;L_0U_1\;=\;0$$

which may also be written as

$$L_0\left(U_1-dU_{|(\bar M,\bar k)}(\,\cdot\,;\mathcal{M}_1,\kappa_1)\right)=(\Omega_1\;+\;\bar k\partial_k c(\bar M,\bar k)\,\kappa_1+c(\bar M,\bar k)\,\kappa_1)\,\bar U'$$

for any $\mathcal{M}_1$. Solvability yields

$$\Omega_1\;+\;\bar k\partial_k c(\bar M,\bar k)\,\kappa_1+c(\bar M,\bar k)\,\kappa_1\;=\;0$$

and with our normalization choices (B.18) reduces to

$$U_1\;=\;dU_{|(\bar M,\bar k)}(\mathcal{M}_1,\kappa_1),\quad \mathcal{M}_1\;=\;\langle U_1\rangle.$$

Compatibility condition $\partial_T\kappa_1=\partial_X\Omega_1$ already gives

$$\text{(B.19)}\qquad \partial_T\kappa_1+\;\partial_X(\bar k\partial_k c_{|(\bar M,\bar k)}\,\kappa_1+c(\bar M,\bar k)\,\kappa_1)\;=\;0.$$

At the next step of the identification, we get with $(\Omega_2,\kappa_2)=(\partial_T\Psi_2,\partial_X\Psi_2)$

$$\begin{aligned}(\Omega_2+c(\bar M,\bar k)\kappa_2)\;\bar U'\;&-\;\kappa_2\,L^{(1)}\bar U'\;-\;L_0U_2\;-\;L^{(1)}\partial_XU_1\\ &-\;(\kappa_1)^2\,L^{(2)}\bar U''\;-\;\kappa_1\,L^{(1)}U_1'\;-\;\partial_X\kappa_1\,L^{(2)}\bar U'\\ &+\;(\Omega_1+c(\bar M,\bar k)\kappa_1)\,U_1'\;+\;\partial_TU_1+c(\bar M,\bar k)\partial_XU_1\\ &+\;\bar k\,\partial_y\left(\frac12 d^2f(\bar U)(U_1,U_1)\right)\;+\;\kappa_1\,\partial_y\left(df(\bar U)\right)\,U_1\;=\;0,\end{aligned}$$

which may also be written

$$\text{(B.20)}\qquad \begin{aligned}L_0\Big(U_2\;&-\;dU_{|(\bar M,\bar k)}(\,\cdot\,;\mathcal{M}_2,\kappa_2)-\frac12 d^2U_{|(\bar M,\bar k)}(\,\cdot\,;(\mathcal{M}_1,\kappa_1),(\mathcal{M}_1,\kappa_1))\Big)\\ &=\;\partial_TU_1+c(\bar M,\bar k)\partial_XU_1-\;\partial_X\kappa_1\,L^{(2)}\bar U'-\;L^{(1)}\partial_XU_1\\ &+\;\left(\Omega_2\;-\;\partial_k\omega(\bar M,\bar k)\,\kappa_2-\frac12 d^2\omega(\bar M,\bar k)\left((\mathcal{M}_1,\kappa_1),(\mathcal{M}_1,\kappa_1)\right)\right)\bar U'\end{aligned}$$

for any $\mathcal{M}_1$. Solvability then reads

$$\text{(B.21)}\quad \partial_T \mathcal{M}_1 \;+\; \partial_X (dF_{|(\bar M,\bar k)}\,(\mathcal{M}_1,\kappa_1)) \;=\; 0,$$

$$\begin{aligned}\text{(B.22)}\qquad \Omega_2 \;&-\; \partial_k\omega(\bar M,\bar k)\,\kappa_2 \;-\; \frac{1}{2}d^2\omega(\bar M,\bar k)\,((\mathcal{M}_1,\kappa_1),(\mathcal{M}_1,\kappa_1))\\ &=\; \partial_X\kappa_1 + \langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_k U_{|(\bar M,\bar k)}\rangle \partial_X\kappa_1 + \langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_M U_{|(\bar M,\bar k)}\rangle \partial_X \mathcal{M}_1.\end{aligned}$$

Note that the latter equation yields

$$\begin{aligned}\text{(B.23)}\qquad &\partial_T\kappa_2 - \partial_X\left(\partial_k\omega(\bar M,\bar k)\,\kappa_2 \;+\; \frac{1}{2}d^2\omega(\bar M,\bar k)\,((\mathcal{M}_1,\kappa_1),(\mathcal{M}_1,\kappa_1))\right)\\ &=\; \partial_X^2\kappa_1 + \partial_X\left(\langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_k U_{|(\bar M,\bar k)}\rangle \partial_X\kappa_1 + \langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_M U_{|(\bar M,\bar k)}\rangle \partial_X \mathcal{M}_1\right).\end{aligned}$$

To proceed, we introduce $\tilde g^k$, $\tilde g^M$, the solutions of

$$\begin{aligned}L_0\;\tilde g^k \;&=\; -\;L^{(1)}\partial_k U_{|(\bar M,\bar k)} - \partial_M U_{|(\bar M,\bar k)}\partial_k F_{|(\bar M,\bar k)}\\ &\quad + \bar U'\langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_k U_{|(\bar M,\bar k)}\rangle - \partial_k U_{|(\bar M,\bar k)}\;\bar k\,\partial_k c_{|(\bar M,\bar k)},\\ L_0\;\tilde g^M \;&=\; -L^{(1)}\partial_M U_{|(\bar M,\bar k)} - \partial_M U_{|(\bar M,\bar k)}(\partial_M F_{|(\bar M,\bar k)} - c_{|(\bar M,\bar k)}\mathrm{Id})\\ &\quad + \bar U'\langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_M U_{|(\bar M,\bar k)}\rangle,\end{aligned}$$

mean free and orthogonal to $u^{adj}(\,\cdot\,;\bar M,\bar k)$ and set $\tilde g = \begin{pmatrix}\tilde g^M & \tilde g^k\end{pmatrix}$. With (B.21) and (B.22), setting $\mathcal{M}_2 = \langle U_2\rangle$, equation (B.20) becomes

$$U_2 \;=\; dU_{|(\bar M,\bar k)}\,(\mathcal{M}_2,\kappa_2) \;+\; \frac{1}{2}d^2U_{|(\bar M,\bar k)}\,(\,\cdot\,;(\mathcal{M}_1,\kappa_1),(\mathcal{M}_1,\kappa_1)) \;+\; \tilde g\;\begin{pmatrix}\partial_X\mathcal{M}_1\\ \partial_X\kappa_1\end{pmatrix}.$$

Finally, substituting (B.2,B.3) into (B.1), and comparing terms of order ε^3, we obtain an equation of the form

$$\partial_T U_2 \;+\; \partial_X\left(df(\bar U)\,U_2 + \frac{1}{2}d^2f(\bar U)(U_1,U_1)\right) \;-\; \partial_X^2 U_1 \;-\; L_0 U_3 \;+\; \partial_y(\;\cdots\;) \;=\; 0,$$

whose solvability implies

$$\begin{aligned}\text{(B.24)}\qquad \partial_T\mathcal{M}_2 \;&+\; \partial_X\left(dF_{|(\bar M,\bar k)}\,(\mathcal{M}_2,\kappa_2) + \frac{1}{2}d^2F_{|(\bar M,\bar k)}\,[(\mathcal{M}_1,\kappa_1),(\mathcal{M}_1,\kappa_1)]\right)\\ &=\; \partial_X^2\mathcal{M}_1 \;-\; \partial_X\left(\langle df(\bar U)\,\tilde g\rangle\;\begin{pmatrix}\partial_X\mathcal{M}_1\\ \partial_X\kappa_1\end{pmatrix}\right).\end{aligned}$$

To write the second order system in a compact form, let us introduce, for arbitrary (M,k),

$$\begin{aligned}\tilde d_{1,1} \;&=\; \mathrm{Id} - \langle df(U(M,k))\,\tilde g^M\rangle, &\tilde d_{1,2} \;&=\; -\langle df(U(M,k))\,g^k\rangle\\ \tilde d_{2,1} \;&=\; \langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_M U_{|(\bar M,\bar k)}\rangle, &\tilde d_{2,2} \;&=\; 1 + \langle u^{adj}(\bar M,\bar k), L^{(1)}\partial_k U_{|(\bar M,\bar k)}\rangle\end{aligned}.$$

With these notations, systems (B.19,B.21), (B.23,B.24) coincide with the first nontrivial systems obtained in the formal expansion of a solution $(\mathcal{M},\kappa)$ of

$$\text{(B.25)}\qquad \left\{\begin{aligned}\partial_t\mathcal{M} \;&+\; \partial_x(F(\mathcal{M},\kappa)) \;&=&\; \partial_x\left(\tilde d_{1,1}\,\partial_x\mathcal{M} + \tilde d_{1,2}\,\partial_x\kappa\right)\\ \partial_t\kappa \;&+\; \partial_x(\kappa\,c(\mathcal{M},\kappa)) \;&=&\; \partial_x\left(\tilde d_{2,1}\,\partial_x\mathcal{M} + \tilde d_{2,2}\,\partial_x\kappa\right)\end{aligned}\right.$$

according to the slow *ansatz*

$$(\mathcal{M},\kappa)(x,t) \;=\; \sum_j \varepsilon^j(\mathcal{M}_j,\kappa_j)(\varepsilon x,\varepsilon t), \qquad (\mathcal{M}_0,\kappa_0) \;=\; (\bar{M},\bar{k}).$$

We call system (B.25), likewise, a *(second-order) Whitham's modulation system.*

As should be clear from the formal derivations, there is some freedom in the choice of the diffusion matrices. This reflects the fact that many systems of conservation laws share the same asymptotic behavior about constant states. We recall next how to classify these systems according to their asymptotic behavior; this will provide a canonical modulation system for our nonlinear analysis.

We emphasize however that the classification below has been so far as we know verified only for symmetrizable hyperbolic-parabolic systems satisfying a Kawashima structural condition [35]. It is unclear to us whether our formally obtained Whitham systems satisfy such conditions. Yet, we know that, by applying to them, on formal grounds, the asymptotic-equivalence reduction, we do obtain a system satisfying such conditions and providing the correct asymptotic behavior. Hence we may safely ignore these technical details.

B.2. **Asymptotic equivalence of systems of conservation laws.**

B.2.1. *General theory.* We now recall the notion of asymptotic equivalence and behavior of solutions of systems of conservation laws near a constant state, useful in our context since, being able to prove modulational behavior, we reduce the dynamics about a periodic wave to motion of parameters near a constant state. Given a general system of conservation laws

$$w_t + (g(w))_x = (B(w)w_x)_x \tag{B.26}$$

and a reference state w_* at which $dg(w_*)$ has distinct eigenvalues, so that $L_* dg(w_*) R_*$ is diagonal for some $L_* = \begin{pmatrix} l_1^* \\ \vdots \\ l_n^* \end{pmatrix}$, $R_* = \begin{pmatrix} r_1^* & \dots & r_n^* \end{pmatrix}$, $L_*R_* = \mathrm{Id}$, define the *quadratic approximant*

$$y_t + A_* y_x + \frac{1}{2}(y^t \Gamma_* y)_x = \tilde{B}_* y_{xx}, \tag{B.27}$$

and the *decoupled quadratic approximant*

$$z_t + A_* z_x + \frac{1}{2}(z^t \tilde{\Gamma}_* z)_x = \tilde{B}_* z_{xx}, \tag{B.28}$$

about w_*, where

$$A_* = dg(w_*), \qquad \Gamma_* := d^2 g(w_*), \qquad \text{and} \qquad B_* := B(w_*), \tag{B.29}$$

$$\tilde{\Gamma}_* := L_*^t \mathrm{diag}\{R_*^t \Gamma_* R_*\} L_*, \qquad \text{and} \qquad \tilde{B}_* := R_* \mathrm{diag}\{L_* B_* R_*\} L_*.^{30} \tag{B.30}$$

Assume the parabolicity condition, $\mathrm{diag}\{L_* B(w_*) R_*\}$ is positive, and define the self-similar nonlinear (resp. linear if $\gamma_j = 0$) diffusion waves $\theta_j(x,t) = t^{-1/2}\bar{\theta}_j(x/\sqrt{t})$ to be the solutions of the Burgers equations (resp. heat equations if $\gamma_j = 0$)

$$\theta_t + \frac{1}{2}(\gamma_j^* \theta^2)_x = \theta_{xx}, \quad \gamma_j^* := [l_j^*(r_j^*)^t \Gamma_* r_j^*]/[l_j^* B_* r_j^*], \tag{B.31}$$

with delta-function initial data $l_j^* m_0\ \delta(\cdot)$, where $m_0 := \int z_0(x)dx$. Then, we have the following fundamental result describing behavior of (B.26)–(B.28) with respect to localized initial perturbations.

[30] Here and elsewhere, we identify as usual bilinear maps with vector-valued matrices, and in particular $d^2 g(w_*)$ with $Hess(g)(w_*)$. Moreover, for these vector-valued matrices, we use coordinate notations so that for instance $\Gamma_*^j \in \mathbb{R}^{n\times n}$ satisfies $w^t \Gamma_*^j w = (w^t \Gamma_* w)_j = d^2 g_j(w,w)$.

Proposition B.1 ([35, 41]). *Let $\eta > 0$. Let w and z be solutions of* (B.26) *and* (B.28) *with initial data w_0 and $z_0 = w_0 - w_*$ such that $E_1 := \|z_0\|_{L^1(\mathbb{R})\cap H^4(\mathbb{R})} + \||\cdot|z_0\|_{L^1(\mathbb{R})}$ is sufficiently small. Then, for $1 \le p \le \infty$, $m_0 := \int_{\mathbb{R}} z_0$, and θ_j as in* (B.31),

$$\|w(t) - w_* - z(t)\|_{L^p(\mathbb{R})} \lesssim E_1(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}+\eta}; \tag{B.32}$$

and

$$\|z(t) - \sum_j \theta_j(\cdot - a_j^*(1+t), b_j^*(1+t))\; r_j^*\|_{L^p(\mathbb{R})} \lesssim E_1(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}+\eta}, \tag{B.33}$$

with $a_j^* := l_j^* A_* r_j^*$, $b_j^* := l_j^* B_* r_j^*$, whence[31]*, if $\eta < 1/4$,*

$$\|w(t) - w_*\|_{L^p(\mathbb{R})},\ \|z(t)\|_{L^p(\mathbb{R})} \gtrsim |m_0|\,(1+t)^{-\frac{1}{2}(1-1/p)}\ .$$

Proposition B.1 asserts that (B.26) and (B.28) (hence also (B.27)) are *asymptotically equivalent* with respect to small localized initial data $w_0 - w_* = z_0 \in L^1(\mathbb{R}, (1+|x|)dx) \cap H^3(\mathbb{R})$, in the sense that the difference between solutions $z(t)$ and $w(t) - w_*$ decays at rate $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}+\eta}$ approximately $(1+t)^{-\frac{1}{4}}$ faster than the (Gaussian) rate $|m_0|\,(1+t)^{-\frac{1}{2}(1-1/p)}$ at which either one typically (i.e., for data with small L^1 first moment) decays. Moreover, through (B.33), it gives a simple description of asymptotic behavior as the *linear superposition of scalar diffusion waves θ_j* moving with characteristic speeds (eigenvalues a_j^*) in the characteristic modes (eigendirections r_j^*) of $dg(w_*)$, satisfying Burgers equations (B.31).

We have also the following more elementary result comparing to the full quadratic approximant.

Proposition B.2 ([26][32]). *Let $\eta > 0$. Let w and y be solutions of* (B.26) *and* (B.27) *with initial data w_0 and $y_0 = w_0 - w_*$ such that $E_0 := \|y_0\|_{L^1(\mathbb{R})\cap H^4(\mathbb{R})}$ is sufficiently small. Then, for $1 \le p \le \infty$,*

$$\|(w - w_* - y)(t)\|_{L^p(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}.$$

An important consequence of Proposition B.2 is that only the quadratic order quantities appearing in (B.27) need be taken into account in the study of asymptotic behavior of (B.26) to the order of approximation considered in Theorem 1.12. Finally, we note the following result following from a proof similar to but much simpler than the one for Proposition B.2 given in [26, Appendix A].

Lemma B.3. *Let k satisfy $k(0) = 0$ and*

$$k_t + ak_x + (\gamma k^2)_x - dk_{xx} = (Fk)_x, \tag{B.34}$$

where a, γ, d are constant, $d > 0$ and F is a given function such that $\|F(t)\|_{L^2(\mathbb{R})} \le E_0(1+t)^{-\frac{1}{4}}$. Then, for any $\eta > 0$, provided E_0 is small enough, for $1 \le p \le \infty$,

$$\|k(t)\|_{L^p(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}.$$

For the sake of completeness, we recall the proof of the previous Proposition in Appendix C.

[31] Computing $\|\theta_j(t)\|_{L^p(\mathbb{R})} = t^{-1/2}\|\bar\theta_j(\cdot/\sqrt{t})\|_{L^p(\mathbb{R})} \sim t^{-\frac{1}{2}(1-1/p)}$.

[32] Though stated in [26, Lemma 1.2] for scalar equations, the proof applies equally to the system case; see Appendix C.

B.2.2. *A first application.* As an immediate application, we may now establish the improved decay bounds (1.22)–(1.21) of Corollary 1.18. We will use these tools again in establishing (1.18).

Proof of Corollary 1.18. Bound (1.21) follows from the assumption $\bar{k}\partial_x h_0 = 0$. For, a solution (M_W, k_W), with an initial data $(*, 0)$, of the decoupled approximating equations (B.28) to (1.7) satisfies $k_W(t) \equiv 0$, since the k equation decouples in (B.28) for the linearly phase-decoupled case. Comparing to the actual solution of (1.7) using (B.32), we obtain the result. Bound (1.22) goes similarly, observing that in the quadratically decoupled case, the k equation in the full quadratic approximating system (B.27) to (1.7), though it does not completely decouple, is of the form (B.34) with $F = \mathcal{O}(M)$. $\square$

Remark B.4. Analogous to (B.34) in the quadratically decoupled case, the rate-determining bound in the linearly decoupled case of Proposition B.1 is the key estimate

$$\|k(t)\|_{L^p} \lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}}$$

established by Liu [40] for quadratic coupling terms involving different modes, thus obeying $k(0) = 0$, $k_t + ak_x + (\gamma k^2)_x - dk_{xx} = (\tilde{\theta}^2)_x$, where $\tilde{\theta}(x,t) = \theta(x - \tilde{a}t, \tilde{b}t)$ with $\tilde{a} \neq a$ and θ a self similar solution of a Burgers equation (B.31). The anomalous rate $(1+t)^{\frac{1}{4}}$ is different from the powers of $(1+t)^{\frac{1}{2}}$ arising in scalar convection–diffusion processes, reflecting the additional complications present in the system case.

B.2.3. *Quadratic approximants of modulation systems.* For later reference, let us write, in the original frame (and not the co-moving one), as

$$\partial_t \begin{pmatrix} M \\ k \end{pmatrix} + \partial_x \begin{pmatrix} dF_{|(\bar{M},\bar{k})}(M,k) \\ d\omega_{|(\bar{M},\bar{k})}(M,k) \end{pmatrix} + \frac{1}{2}\partial_x \begin{pmatrix} d^2F_{|(\bar{M},\bar{k})}(M,k) \\ d^2\omega_{|(\bar{M},\bar{k})}((M,k),(M,k)) \end{pmatrix} = \tilde{B}_*\partial_x^2 \begin{pmatrix} M \\ k \end{pmatrix} \tag{B.35}$$

the quadratic approximant of (1.7) (obtained as (B.16) and (B.25) above). As pointed out in Remark 4.11, it follows from Lemma 4.10 that this system is independent of the choices made in the course of the formal derivation.

From the general theory, we know that instead of comparing $(\mathcal{M}, \kappa)$ in Theorem 1.12 to a solution $(\mathcal{M}_W, \kappa_W)$ of (1.7), we only need to compare it with $(\bar{M}, \bar{k}) + (M_W, k_W)$ with (M_W, k_W) a solution of (B.35) expressed in the co-moving frame.

B.3. **Implicit change of variables.** Our nonlinear analysis begins with an implicit nonlinear change of variable (2.5). We explain now how the modulation system is affected by this change of variables. We could have first performed this implicit change of variables then carried out the formal modulation process, but we find more enlightening to change the system *a posteriori*.

Since our diffeomorphism is close to identity, only nonlinear terms should be changed, and from the asymptotic equivalence theory we know that nonlinear terms are relevant only in the hyperbolic part. Therefore it is enough to investigate how (B.5,B.8) is altered. Let us introduce Φ_0 such that $\Phi_0(\Psi_0(X,T),T) = X$. Recall that $\partial_T\Psi_0 = \omega(\mathcal{M}_0, \partial_X\Psi_0)$. Therefore if A, B are such that $\partial_T A + \partial_X B = 0$ then $(\tilde{A}, \tilde{B})(X,T) = (A,B)(\Phi_0(X,T),T)$ implies

$$\partial_T\tilde{A} - \frac{\partial_T\Phi_0}{\partial_X\Phi_0}\,\partial_X\tilde{A} + \frac{1}{\partial_X\Phi_0}\,\partial_X\tilde{B} = 0$$

also written $\partial_T\left(\partial_X\Phi_0\tilde{A}\right) + \partial_X\left(\tilde{B} - \partial_T\Phi_0\tilde{A}\right) = 0$ or

$$\partial_T\left(\partial_X\Phi_0\tilde{A}\right) + \partial_X\left(\tilde{B} - c\left(\tilde{\mathcal{M}}_0, \frac{1}{\partial_X\Phi_0}\right)\tilde{A}\right) = 0$$

with $\tilde{\mathcal{M}}_0(X,T) = \mathcal{M}_0(\Phi_0(X,T),T)$. Note that this kind of manipulation is completely similar to the ones needed to perform usual Lagrangian change of coordinates and of course closely related to the computations involved in the proof of Lemma 2.3. As expected, applying this to (B.5) leads to a trivial equation while an application on the trivial equation $\partial_T(1)+\partial_x(0)=0$ gives

$$\partial_T\left(\partial_X\Phi_0\right)\;-\;\partial_X\left(c\left(\tilde{\mathcal{M}}_0,\frac{1}{\partial_X\Phi_0}\right)\right)\;=\;0.$$

Equation (B.5) is changed into

$$\partial_T\left(\partial_X\Phi_0\tilde{\mathcal{M}}_0\right)\;+\;\partial_X\left(F\left(\tilde{\mathcal{M}}_0,\frac{1}{\partial_X\Phi_0}\right)-c\left(\tilde{\mathcal{M}}_0,\frac{1}{\partial_X\Phi_0}\right)\tilde{\mathcal{M}}_0\right)\;=\;0.$$

At the hyperbolic level, we are thus lead to the system

$$\begin{aligned}\partial_T p\;&-\;\partial_X\left(c\left(\tfrac{\mathcal{M}}{p},\tfrac{1}{p}\right)\right)\;=\;0\\ \partial_T\mathcal{M}\;&+\;\partial_X\left(F\left(\tfrac{\mathcal{M}}{p},\tfrac{1}{p}\right)-c\left(\tfrac{\mathcal{M}}{p},\tfrac{1}{p}\right)\tfrac{\mathcal{M}}{p}\right)\;=\;0\end{aligned}$$

whose quadratic expansion in

$$(p,\mathcal{M})\;=\;\left(\frac{1}{\bar{k}},\frac{\bar{M}}{\bar{k}}\right)+\left(\frac{-k}{\bar{k}}\frac{1}{\bar{k}},\frac{-k}{\bar{k}}\frac{\bar{M}}{\bar{k}}+\frac{M}{\bar{k}}\right)$$

gives

$$\begin{aligned}\partial_T k-\bar{k}\,c(\bar{M},\bar{k})\partial_X k\;&-\;\bar{k}\partial_X\left(d\omega_{|(\bar{M},\bar{k})}(M,k)+\tfrac{1}{2}d^2\omega_{|(\bar{M},\bar{k})}((M,k),(M,k))\right)\;=\;0\\ \partial_T M-\bar{k}\,c(\bar{M},\bar{k})\partial_X M\;&+\;\partial_X\left(dF_{|(\bar{M},\bar{k})}(M,k)+\tfrac{1}{2}d^2F_{|(\bar{M},\bar{k})}((M,k),(M,k))\right)\\ &+\;\bar{k}\partial_X\left(\frac{k}{\bar{k}}\left(dF_{|(\bar{M},\bar{k})}(M,k)-c(\bar{M},\bar{k})M\right)-dc_{|(\bar{M},\bar{k})}(M,k)M\right)\;=\;0.\end{aligned}$$

Two main comments are in order: 1. We end up naturally with equations expressed in a co-moving frame thus no further change is needed. 2. The wavenumber equation remains unaltered at this level of description. This explains why the fact that the implicit change of variables could change the modulation equations was not revealed by previous studies [26, 57] focusing on situations where no other wave parameter is involved,

Remark B.5. Though we do not need it for the present semilinear analysis, let us describe for the sake of generality what would happen for a full quasilinear parabolic system. For

$$\partial_t\begin{pmatrix}\mathcal{M}\\ \Psi_x\end{pmatrix}\;+\;\partial_x A(\mathcal{M},\Psi_x)\;=\;\partial_x\left(D(\mathcal{M},\Psi_x)\partial_x\begin{pmatrix}\mathcal{M}\\ \Psi_x\end{pmatrix}\right)$$

with

$$\partial_t\Psi\;+\;A_{n+1}(\mathcal{M},\Psi_x)\;=\;D_{n+1}(\mathcal{M},\Psi_x)\partial_x\begin{pmatrix}\mathcal{M}\\ \Psi_x\end{pmatrix}$$

where $A_{n+1}=e_{n+1}\cdot A$, $D_{n+1}=e_{n+1}\cdot D$, the transformation $\Phi=\Psi^{-1}$, $\tilde{\mathcal{M}}=\mathcal{M}\circ\Phi$ leads to

$$\partial_t\Phi\;-\;\Phi_x\,A_{n+1}(\tilde{\mathcal{M}},1/\Phi_x)\;=\;-D_{n+1}(\tilde{\mathcal{M}},1/\Phi_x)\partial_x\begin{pmatrix}\tilde{\mathcal{M}}\\ 1/\Phi_x\end{pmatrix}$$

with

$$\begin{aligned}\partial_t\tilde{\mathcal{M}}\;&+\;\partial_x\left(A_\perp(\tilde{\mathcal{M}},1/\Phi_x)-\partial_x\Phi\,A_{n+1}(\tilde{\mathcal{M}},1/\Phi_x)\,\tilde{\mathcal{M}}\right)\\ &=\;\partial_x\left(\tfrac{1}{\partial_x\Phi}\,D_\perp(\tilde{\mathcal{M}},1/\Phi_x)\partial_x\begin{pmatrix}\tilde{\mathcal{M}}\\ 1/\Phi_x\end{pmatrix}-D_{n+1}(\tilde{\mathcal{M}},1/\Phi_x)\partial_x\begin{pmatrix}\tilde{\mathcal{M}}\\ 1/\Phi_x\end{pmatrix}\tilde{\mathcal{M}}\right)\\ \partial_t(\partial_x\Phi)\;&-\;\partial_x\left(\Phi_x\,A_{n+1}(\tilde{\mathcal{M}},1/\Phi_x)\right)\;=\;-\partial_x\left(D_{n+1}(\tilde{\mathcal{M}},1/\Phi_x)\partial_x\begin{pmatrix}\tilde{\mathcal{M}}\\ 1/\Phi_x\end{pmatrix}\right)\end{aligned}$$

where $A_\perp = \begin{pmatrix} & 0 \\ \mathrm{Id}_{d\times d} & \vdots \\ & 0 \end{pmatrix} A$, $D_\perp = \begin{pmatrix} & 0 \\ \mathrm{Id}_{d\times d} & \vdots \\ & 0 \end{pmatrix} D$.

Collecting the results of this appendix, we find that to validate the formal Whitham modulation approximation, we only need to compare the couple $(M, \bar{k}\psi_x)$ of Theorem 1.12 to a solution (M_W, k_W) of

$$\partial_t \begin{pmatrix} M \\ k \end{pmatrix} + \bar{k}A_*\partial_x \begin{pmatrix} M \\ k \end{pmatrix} + \frac{1}{2}\bar{k}\partial_x \begin{pmatrix} d^2F_{|(\bar{M},\bar{k})}((M,k),(M,k)) \\ d^2\omega_{|(\bar{M},\bar{k})}((M,k),(M,k)) \end{pmatrix} - \bar{k}^2\tilde{B}_*\partial_x^2 \begin{pmatrix} M \\ k \end{pmatrix}$$
$$+ \bar{k}\partial_x \begin{pmatrix} \frac{k}{\bar{k}}\left(dF_{|(\bar{M},\bar{k})}(M,k) - c(\bar{M},\bar{k})M\right) - dc_{|(\bar{M},\bar{k})}(M,k)M \\ 0 \end{pmatrix} = 0,$$

where $A_* = \partial_{(M,k)}(F - \bar{c}M, -\omega - \bar{c}k)|_{(\bar{M},\bar{k})}$. For writing convenience, we denote this system by

$$\partial_t \begin{pmatrix} M \\ k \end{pmatrix} + \bar{k}A_*\partial_x \begin{pmatrix} M \\ k \end{pmatrix} - \partial_x\Big(\frac{1}{2}\begin{pmatrix} M \\ k \end{pmatrix}^T \Gamma_* \begin{pmatrix} M \\ k \end{pmatrix}\Big) = \bar{k}^2\tilde{B}_*\partial_x^2 \begin{pmatrix} M \\ k \end{pmatrix}. \tag{B.36}$$

Likewise, ψ in Theorem 1.12 needs then to be compared with ψ_W a solution of

$$\partial_t\psi + e_{n+1}\cdot A_* \begin{pmatrix} M_W \\ k_W \end{pmatrix} - \frac{1}{\bar{k}}e_{n+1}\cdot\Big(\frac{1}{2}\begin{pmatrix} M_W \\ k_W \end{pmatrix}^T \Gamma_* \begin{pmatrix} M_W \\ k_W \end{pmatrix}\Big) = \bar{k}e_{n+1}\cdot\tilde{B}_*\partial_x \begin{pmatrix} M_W \\ k_W \end{pmatrix}. \tag{B.37}$$

Appendix C. Asymptotic equivalence of quadratic approximants

For completeness, we include here a proof of Proposition B.2 including the treatment of off-diagonal diffusion terms not arising in the scalar case considered in [25].

Proof of Proposition B.2. (*Case $B_* = \tilde{B}_*$.*) We first review the case $B_* = \tilde{B}_*$ treated in [25]. By the general results of [35], provided $E_0 := \|y_0\|_{L^1\cap H^3(\mathbb{R})}$ is sufficiently small, we have for $1 \le p \le \infty$

$$\|w(t) - w_*\|_{L^p(\mathbb{R})},\ \|y(t)\|_{L^p(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})},$$
$$\|w_x(t)\|_{H^1(\mathbb{R})},\ \|y_x(t)\|_{H^1(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{3}{4}}.$$

Setting $\delta := w_* + y - w$, we have, subtracting and rearranging,

$$\delta_t + A_*\delta_x - B_*\delta_{xx} = \partial_x\mathcal{F}, \qquad \mathcal{F} = \mathcal{O}((|w-w_*| + |y|)\delta) + \mathcal{O}(|w-w_*|^3) + \mathcal{O}(|w-w_*||w_x|),$$

with $\delta|_{t=0} = 0$ and A_* and B_* as in (B.29)–(B.30). By Duhamel's formula,

$$\delta(t) = \int_0^t \sigma(t-s)\partial_x\mathcal{F}(s)ds,$$

where σ is the solution operator of the parabolic system of conservation laws $u_t + A_*u_x - B_*u_{xx} = 0$. Applying the standard bounds [41] $\|\sigma(t)\partial_x^r h\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{r}{2}}\|h\|_{L^q(\mathbb{R})}$, $1 \le q \le p \le \infty$, together with

$$\|\mathcal{F}(t)\|_{L^q(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{2}(1-1/q)-\frac{1}{4}}(\|\delta(t)\|_{L^2(\mathbb{R})} + \|w_x(t)\|_{L^2(\mathbb{R})}) + E_0^2(1+t)^{-\frac{1}{2}(1-1/q)-1},$$

$1 \le q \le 2$, we find, defining $\nu(t) := \sup_{0\le s\le t} \|\delta(s)\|_{L^2(\mathbb{R})}(1+s)^{\frac12(1-1/p)+\frac12-\eta}$, that, for all $1 \le p \le \infty$,

$$\begin{aligned}
\|\delta(t)\|_{L^p(\mathbb{R})} &\lesssim \int_0^{t/2} (t-s)^{-\frac12(1-1/p)-\frac12}\|\mathcal{F}(s)\|_{L^1(\mathbb{R})}ds \\
&\quad + \int_{t/2}^t (t-s)^{-\frac12(1/(\min(2,p))-1/p)-\frac12}\|\mathcal{F}(s)\|_{L^{\min(2,p)}(\mathbb{R})}ds \\
&\lesssim \int_0^{t/2} (t-s)^{-\frac12(1-1/p)-\frac12}(\nu(t)E_0+E_0^2)(1+s)^{-1+\eta}ds \\
&+ \int_{t/2}^t (t-s)^{-\frac12(1/(\min(2,p))-1/p)-\frac12}(\nu(t)E_0+E_0^2)(1+s)^{-1+\eta-\frac12(1-1/(\min(2,p)))}ds \\
&\lesssim E_0(E_0+\nu(t))(1+t)^{-\frac12(1-1/p)-\frac12+\eta},
\end{aligned}$$

whence $\nu(t) \le C_\eta E_0\,(E_0+\nu(t))$. This implies that $\nu(t) \le 2C_\eta E_0^2$ for $E_0 < 1/(2C_\eta)$, giving

$$\|\delta(t)\|_{L^p(\mathbb{R})} \le 2C_\eta E_0^2(1+t)^{-\frac12(1-1/p)-\frac12+\eta}, \qquad 1 \le p \le \infty.$$

(*General case.*) We treat now the general case that $B(w) = B_* + \mathcal{O}(w-w_*)$ with B_* constant but not equal to $\tilde B_*$. Defining again $\delta := w_* + y - w$, and denoting by $\tilde\sigma(t)$ the solution operator of linear system $u_t + A_*u_x - \tilde B_* u_{xx} = 0$, we have by Duhamel's principle

$$\begin{aligned}
\delta(t) \;&=\; (\tilde\sigma-\sigma)(t)w_0 + \textstyle\int_0^t (\tilde\sigma-\sigma)(t-s)\partial_x\mathcal{O}(|w-w_*|^2)(s)ds \\
&\quad + \textstyle\int_0^t \tilde\sigma(t-s)\partial_x\mathcal{O}(|\delta|(|w-w_*|+|y|))(s)ds \\
&\quad + \textstyle\int_0^t \sigma(t-s)\partial_x\mathcal{O}(|w-w_*|^3+|w-w_*||w_x|)(s)ds.
\end{aligned}$$

From [35], provided $E_0 := \|y_0\|_{L^1\cap H^4(\mathbb{R})}$ is sufficiently small, we have for $1 \le p \le \infty$

$$\begin{aligned}
\|w(t)-w_*\|_{L^p(\mathbb{R})},\ \|y(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac12(1-\frac1p)}, \\
\|w_x(t)\|_{H^2(\mathbb{R})},\ \|y_x(t)\|_{H^2(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac34}.
\end{aligned}$$

Applying the bounds [35, 41] $\|(\tilde\sigma-\sigma)(t)\partial_x^r h\|_{L^p(\mathbb{R})} \lesssim t^{-\frac12(\frac1q-\frac1p)-\frac r2}(1+t)^{-\frac12}\|h\|_{L^q(\mathbb{R})} + e^{-\theta t}\|\partial_x^r h\|_{L^p(\mathbb{R})}$, $1 \le q \le p \le \infty$ for $r = 0, 1$ (and some $\theta > 0$), estimating

$$\begin{aligned}
&\Big\|\int_0^t (\tilde\sigma-\sigma)(t-s)\partial_x\mathcal{O}(|w-w_*|^2)(s)ds\Big\|_{L^p(\mathbb{R})} \\
&\quad\lesssim \int_0^{t/2}(t-s)^{-\frac12(1-1/p)-1}\||w-w_*|^2(s)\|_{L^1(\mathbb{R})}ds + \int_0^t e^{-\theta(t-s)}\||w-w_*|^2(s)\|_{L^p(\mathbb{R})}ds \\
&\qquad + \int_{t/2}^t (t-s)^{-\frac12}(1+t-s)^{-\frac12}\||w-w_*|^2(s)\|_{L^p(\mathbb{R})}ds \\
&\quad\lesssim E_0\int_0^{t/2}(t-s)^{-\frac12(1-1/p)-1}(1+s)^{-\frac12}ds + E_0\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac12(1-1/p)-\frac12}ds \\
&\qquad + E_0\int_{t/2}^t (t-s)^{-\frac12}(1+t-s)^{-\frac12}(1+s)^{-\frac12(1-1/p)-\frac12}ds \\
&\quad\lesssim E_0(1+t)^{-\frac12(1-1/p)-\frac12}\log(2+t),
\end{aligned}$$

and other terms either similarly or similarly as in the previous case, we obtain the result. $\square$

Appendix D. Generalizations

We conclude in this appendix by describing briefly extensions to more general types of equations arising in applications, and the modifications in our arguments that are needed to accomplish this, discussing also, when possible, the verification of (H1)–(H3) and (D1)–(D3) in specific cases.

D.1. **Extensions in type: quasilinear and partially parabolic systems.** Our analysis carries over in straightforward fashion to divergence-form systems of general quasilinear $2r$-parabolic type. For example, the spectral preparation results of Lemma 1.5, Proposition 1.7, and Proposition 3.1 all go through essentially as written, depending on no special structure other than divergence form. From these low-frequency/Bloch number descriptions, we obtain the same linear bounds on the critical modes s^{p} as described here in the 2-parabolic semilinear case. The high-frequency and or high Bloch number analysis also go through unchanged, the former depending again only on the spectral preparation results and the latter depending only (through Prüss' Theorem) on high-frequency resolvent bounds following from (but not requiring) sectoriality of the linearized operator L about the wave. This completes the linear analysis.

Likewise, by Remark 2.4, we obtain the useful representation (2.6) of the nonlinear perturbation equations stated in Lemma 2.3, with sources $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ of quadratic order in v, ψ_x, ψ_t, and a finite number of their derivatives, which was all that was needed for our nonlinear arguments. To obtain the nonlinear damping estimate of Proposition 2.5, we note that (2.14) becomes

$$(1-\psi_x)v_t + (-1)^r\bar{k}^{2r}\partial_x(B(\tilde{U},\dots,\partial_x^{2r-2}\tilde{U})\partial_x^{2r-1}v) = \text{lower order terms},$$

$\tilde{U} = \bar{U} + v$. Thus, taking the $L^2(\mathbb{R})$ inner product against $\sum_{j=0}^K \dfrac{(-1)^j\partial_x^{2j}v}{1-\psi_x}$, integrating by parts, and rearranging, we obtain $\frac{d}{dt}\|v\|^2_{H^K(\mathbb{R})}(t) \leq -\tilde{\theta}\|\partial_x^{K+r}v(t)\|^2_{L^2(\mathbb{R})}$ + lower order terms, similarly as in the second-order semilinear case, leading thereby to

$$\frac{d}{dt}\|v\|^2_{H^K(\mathbb{R})}(t) \leq -\theta\|v(t)\|^2_{H^K(\mathbb{R})} + C\left(\|v(t)\|^2_{L^2(\mathbb{R})} + \|(\psi_t,\psi_x)(t)\|^2_{H^K(\mathbb{R})}\right)$$

and (by Gronwall's inequality) the result. See the proof of [5, Proposition 3.4], for full details in the fourth-order semilinear case.

Combining these ingredients, we obtain, modulo an appropriate increase in the integer K encoding regularity requirements, stability, as stated in Theorem 1.10, and refined stability, as stated in Proposition 4.5, yielding the first part (1.16) of description of asymptotic behavior in Theorem 1.12. By Remark 4.15, we get also a partial version of the second part (1.17)–(1.19) of Theorem 1.12, but describing comparisons not to the Whitham system, but only to a second-order hyperbolic-parabolic system agreeing with the Whitham system in its linearization about the constant state $(\bar{M},\bar{k})$. This in turn yields the conclusions of (1.21), Corollary 1.18, regarding decay with respect to localized perturbations for linearly phase-decoupled systems.

Finally, to recover the full result (1.17)–(1.19) of Theorem 1.12, comparing to the exact Whitham system, and thus the sharpened decay rate (1.22) for localized data in the quadratically decoupled case, we have only to observe that performing the same computations as in Appendix A (differentiating the traveling-wave ODE), and in the proof of Lemma 4.9 (pulling out quadratic order parts of nonlinear term $\mathcal{N}$) while carrying along the additional higher-order terms arising in the general case, we obtain a higher-order analog of Lemma 4.14, expressing the resulting quadratic coupling constants (means) in terms of derivatives of first-order terms arising in the Whitham system, after which computations go as before to yield the result; see the proof of Theorem 1.12, Section 4.6.

This completes the treatment of the quasilinear $2r$-parabolic case. Reviewing the above discussion, but omitting algebraic considerations on which we focus in the next section, we find that the two ingredients needed to treat more general divergence-form systems are the nonlinear damping estimate used to control higher-derivative by lower-derivative norms, and the high-frequency linearized resolvent bounds used to apply Prüss' Theorem. For, these were the only two places where we used the parabolic form of the equations; the rest of the argument was completely general, Moreover, the second, linearized, estimate can typically be obtained by a linearized version of the same energy estimate that is used to obtain the first, damping-type estimate. This allows us, in particular, to treat (partially parabolic) symmetric hyperbolic–parabolic equations such as arise in continuum mechanics, using "Kawashima-type" energy estimates as described in [35], and variants thereof. See, for example, Proposition 4.4 (proved in Appendix A) and Lemma B.1 in [33].

Remark D.1. The strategy of using a common energy estimate to get, simultaneously, damping high-frequency resolvent, and high-frequency decay estimates, with derivative gains in the first compensating for derivative losses in the third, originates in the study of viscous shock stability; see [76, Section 4.2.1]. For simpler, and somewhat sharpened, versions in this context, see [37, 45].

D.2. **Extensions in form: an abstract continuum of models.** Still more generally, we may treat the full class of systems

$$u_t + f(u)_x = g(u) + (B^1(u)u_x)_x + (B^2(u,u_x)u_{xx})_x + \dots, \tag{D.1}$$

$u, f, g \in \mathbb{R}^n$, $B_j \in \mathbb{R}^{n\times n}$, with $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $g = \begin{pmatrix} 0 \\ g_2 \end{pmatrix}$, $B^j = \begin{pmatrix} B^j_{11} & B^j_{12} \\ B^j_{21} & B^j_{22} \end{pmatrix}$, $u_2 \in \mathbb{R}^r$, including both divergence- and nondivergence-type equations. Note that this includes both reaction diffusion and conservation law cases as limits $f \equiv 0$ and $g \equiv 0$, but also many cases in between: for example, the viscous relaxation case $n = 2$, $r = 1$ occurring for the Saint-Venant equations (1.28), or the case $n = 3$, $r = 1$ occurring for the Bénard–Marangoni model (D.4) below.

For such models, integrating the conservative u_1 equation in the traveling-wave ODE, and writing as an $N \times N$ first-order system, we obtain from the requirement of periodicity N constraints, while we have $N + n - r + 2$ degrees of freedom consisting of the initial condition $u(0)$, the wave number $k \in \mathbb{R}$, the speed $c \in \mathbb{R}$, and the constant of integration $q_1 \in \mathbb{R}^{n-r}$ arising from integration of the u_1 equation; thus, we expect generically a manifold of periodic solutions of dimension $n - r + 2$. In the reaction-diffusion case $r = n$, this returns the familiar value 2, or, up to translation, a one-dimensional family (generically) indexed by wave number k. In the conservation law case $r = 0$, it returns the value $n+2$, leading, up to translation, to an $(n+1)$-dimensional family as in hypothesis (H2) of the introduction.

Substituting this value $n-r+1$ in hypotheses (H2) and (D3), therefore, we readily obtain by the same derivation as for (1.7) a modified Whitham system consisting of the $(n - r + 1) \times (n - r + 1)$ system of viscous conservation laws

$$\begin{aligned} \mathcal{M}_t + \bar{k}(F - \bar{c}\mathcal{M})_x &= \bar{k}^2(d_{11}\mathcal{M}_x + d_{12}\kappa_x)_x, \\ \kappa_t + \bar{k}(-\omega - \bar{c}\kappa)_x &= \bar{k}^2(d_{21}\mathcal{M}_x + d_{22}\kappa_x)_x, \end{aligned} \tag{D.2}$$

where $\omega(\mathcal{M}, \kappa) = -\kappa c(\mathcal{M}, \kappa)$ denotes time frequency, $\mathcal{M} := \int_0^1 U_1^{\mathcal{M},\kappa}(x)dx$ and

$$F(\mathcal{M}, \kappa) := \int_0^1 \big(f_1(U^{\mathcal{M},\kappa}(x)) - \sum_j (B^j_{11}, B^j_{12})(U^{\mathcal{M},\kappa,\dots}(x))(U^{\mathcal{M},\kappa})'(x))\big)\, dx$$

denote mean and mean "total flux" in the u_1 coordinate, and $d_{ij}(\mathcal{M}, \kappa)$ are determined by higher-order corrections. (Note that, for $B^j \equiv$ constant, the terms involving B^j are perfect derivatives, so disappear; this explains the fact that they were not present in the discussion of the second-order semilinear case.)

Likewise, we obtain in straightforward fashion analogs of the spectral preparation results of Lemma 1.5, Proposition 1.7, and Proposition 3.1, thus yielding corresponding linear bounds on critical modes $s^{\rm p}$. Note that the slow decay rates that may arise at the linear level from a possible Jordan block will still be compensated by the special structure of the nonlinear terms, coming now in the form

$$\mathcal{N} \;=\; \partial_t \mathcal{N}_0 + \partial_x \mathcal{N}_1 \;+\; \begin{pmatrix} 0_{(n-r)\times(n-r)} & 0_{(n-r)\times r} \\ 0_{r\times(n-r)} & \mathrm{Id}_{r\times r} \end{pmatrix} \mathcal{N}_2.$$

See for example [48] for a careful derivation of the second-order derivative Whitham system up to linear and quadratic order in first-order derivative terms, and [33] for a proof of the needed spectral preparation results in the Saint-Venant case (1.28). Indeed, so long as the nonlinear structure of the equations permits a nonlinear damping estimate as in Proposition 2.5, and high-frequency linearized resolvent estimates as needed to apply Prüss' Theorem in estimating high-frequency linearized behavior as in the proof of (3.18) and (3.19)) above, we obtain again (modulo increase in the exponent of regularity K) the stability results of Theorem 1.10 and Proposition 4.5, and a partial version of Theorem 1.12 describing comparisons to a second-order hyperbolic-parabolic system agreeing with (D.2) in its linearization about the constant state $(\bar{M}, \bar{k})$., yielding again the result (1.21), of Corollary 1.18 asserting decay with respect to localized perturbations for linearly phase-decoupled systems.

That is, we obtain in this case exactly the conclusions cited in the examples of the introduction, obtained by examination of the linearization of the first-order part of the Whitham equations.

To recover the full result (1.17)–(1.19) of Theorem 1.12 showing convergence to the exact Whitham system, one also needs an analog of Lemma 4.14. But, the only difference between the (formal) computations of the derivation in Subsection B.1.2 and the ones of Lemma 4.14 is that the former are carried out before the implicit change of variables, while the latter are carried out after. Thus, analogs of Lemma 4.14 essentially follow by commutation of an implicit change of variables and expansions to a desired order.

D.3. **Verification of (H1)–(H3), (D1)–(D3).** Regarding verification of our stability hypotheses, we recall that, assuming the trivial regularity hypothesis (H1), hypothesis (H2) is implied by (D1)–(D3), by Lemma 1.6, while (H3) by Proposition 1.7 can generally be verified by the same spectral expansion process needed to verify (D2). Meanwhile, (D1)–(D3) can be verified numerically by Galerkin approximation or numerical Evans function computation (see, e.g., [18, 5, 6]), or, in some cases, analytically, using bifurcation theory (see, e.g., [60]) or singular perturbations (see, e.g., [27]). For general discussion, see [3, 33, 6].

D.4. **Applications revisited.** We now discuss previous examples and some new ones in a bit more depth.

D.4.1. *The Korteweg-de Vries/Kuramoto-Sivashinsky equation.* A more canonical form of (1.26) is

$$u_t + \gamma\partial_x^4 u + \varepsilon\partial_x^3 u + \delta\partial_x^2 u + \partial_x f(u) = 0, \quad \gamma, \delta > 0,$$

modeling phenomena from plasma and flame-front instabilities to inclined thin-film flow [38, 63, 54, 48]. As a fourth-order parabolic equation, this fits the framework of Section D.1, so that all of the results of this paper apply. Spectral stability has been studied in detail in [12, 5], indicating the existence of both spectrally stable and unstable waves; in particular, (H1)–(H3) and (D1)–(D3) have been shown in [5] to hold for a wide variety of waves. We note that stability under these hypotheses has been proven for localized perturbations in [5]; the new observations here are asymptotic behavior, and decay for nonlocalized perturbations.

D.4.2. *The Saint-Venant equations.* Recall, in Lagrangian coordinates, the Saint-Venant equations

$$\begin{aligned} &\tau_t - u_x = 0, \\ &u_t + ((2F)^{-1}\tau^{-2})_x = 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x, \end{aligned} \tag{D.3}$$

where $\tau := h^{-1}$, h is fluid height, u is fluid velocity, and x is a Lagrangian marker. These are not parabolic, yet nonlinear damping and high-frequency resolvent estimates can still be carried out, yielding by the discussion of Section D.2 all of the results of this paper. Specifically, nonlinear damping is established in [33, Proposition 4.4] (proved in Appendix A of the reference), under the "slope condition" $\nu \bar{u}_x < F^{-1}$, where $\bar{U} = (\bar{\tau}, \bar{u})$, a technical condition that appears to hold in most cases of interest, but which we expect can be dropped.[33] Again, the new observation here is asymptotic behavior, and also stability under nonlocalized perturbations, stability under localized data having been established in [33]. As noted earlier, (D.3) is a balance law rather than a conservation law, with nonconservative source term g.

D.4.3. *The capillary Saint-Venant equations.* With capillary pressure effects, (D.3) becomes

$$\begin{aligned} \tau_t - u_x &= 0, \\ u_t + ((2F)^{-1}\tau^{-2})_x &= 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x - \sigma(\tau^{-5}\tau_{xx} - \frac{5}{2}\,\tau^{-6}(\tau_x)^2)_x \end{aligned}$$

where $\sigma > 0$ is the coefficient of capillarity. These equations can be reduced by Kotschote's [36] method of auxiliary variables (introducing $z := \tau_x$) to a 3×3 second-order quasilinear parabolic system

$$\begin{aligned} \tau_t - u_x + z_x &= \tau_{xx}, \\ z_t &= u_{xx}, \\ u_t + ((2F)^{-1}\tau^{-2})_x &= 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x - \sigma(\tau^{-5}z_x - \frac{5}{2}\,\tau^{-6}z^2)_x \end{aligned}$$

to which standard techniques can be applied [69, 74]. This fits the framework of Section D.1, yielding all of the results of this paper, the only change being in the regularity assumptions on data, which must be incremented by one to accommodate the new variable $z = \tau_x$. Existence and spectral stability or instability of these waves is a topic of ongoing investigation [2].

D.4.4. *Bénard–Marangoni flow.* A qualitative model introduced in [20] for Bénard–Marangoni flow, or flow driven by temperature-induced surface tension variation, is

$$\begin{aligned} u_t &= -(1 + u_{xx})_{xx} + \varepsilon^2 u + f(u, v_x, w_x), \\ v_t &= v_{xx} + v_x + g_1(u, v, w)_x, \\ w_t &= w_{xx} - w_x + g_2(u, v, w)_x, \end{aligned} \tag{D.4}$$

with $f(u, v_x, w_x) = -u^3 + \gamma(uv_x + uw_x)$, $g_1(u, v, w) = -uv$, $g_2(u, v, w) = -uw$. Though of mixed fourth-order parabolic/second-order parabolic form, it is readily seen that these equations are both sectorial and admit a nonlinear damping estimate; moreover, they are of the mixed conservative/nonconservative form (D.1). Thus, by the discussion of Sections D.1 and D.2, the main results of this paper apply, giving stability and behavior in terms of a 3×3 hyperbolic–parabolic system agreeing with the Whitham system (D.2).

Let us now discuss existence, the form of the Whitham equations, and validation of (D1)–(D3). Setting $v \equiv w \equiv 0$, we find that the equations reduce to the Swift–Hohenberg equation (1.23) for u, with bifurcation parameter $r = \varepsilon^2$ restricted to the positive side of the bifurcation point $r = 0$ at which periodic solutions appear. Thus, we inherit from the Swift–Hohenberg equations a special class of periodic solutions with (v, w) vanishing. Up to translation, such solutions are given by the 2-parameter family of zero-speed $\frac{2\pi}{1+\varepsilon\omega}$-periodic Swift–Hohenberg solutions

$$\bar{U}^{\omega,\varepsilon}(x) = \frac{2\varepsilon(\sqrt{1 - 4\omega^2}}{\sqrt{3}} \cos((1 + \varepsilon\omega)x) + \mathcal{O}(\varepsilon^2), \quad (\bar{V}^{\omega,\varepsilon}, \bar{W}^{\omega,\varepsilon})(x) \equiv (0, 0), \tag{D.5}$$

[33] Linearized analysis suggests that the sharp condition is, rather, some averaged version of this one, which holds trivially by the fact that perfect derivatives have zero mean [6].

where ε, recall, is the bifurcation parameter, a fixed constant in (D.4). However, there are many other solutions for which $(\bar{V}, \bar{W}) \not\equiv (0,0)$, yielding an additional two parameters in the description of nearby periodic traveling waves. Moreover, though the Swift–Hohenberg solutions are zero speed as a result of reflection symmetry (see Remark 1.27), reflection symmetry of (D.4) is broken as soon as $(v, w) \not\equiv (0,0)$, and so in general these waves may have arbitrary speed. It is our expectation, therefore, that the Whitham system is not phase-decoupled even about such special waves.

Numerical experiment by Galerkin approximation in [70] indicate that solutions (D.5) satisfy stability conditions (D1)–(D3) for $\omega = 0$ and $\varepsilon > 0$ in a moderate range. Here, we demonstrate the same conclusion for $|\omega| < 1/2\sqrt{3}$ and $\varepsilon << 1$, using decoupling of the equations and known analytical results for the Swift–Hohenberg equation, at the same time obtaining the limiting $\varepsilon \to 0$ coefficients of the linearized Whitham system about $(\bar{M}, \bar{k}) = (0, \bar{k})$. It would be interesting to carry out a systematic numerical stability investigation as in [4, 5] on the entire parameter range, and in particular to determine phase-coupling or decoupling of the associated Whitham system.

Proof of (D1)–(D3). About the special solutions (D.5), the linearized eigenvalue equations are

$$\begin{aligned} \lambda u &= L^0 u + Mv + Nw, \\ \lambda v &= L^+ v, \\ \lambda w &= L^- w, \end{aligned} \tag{D.6}$$

where L_0 is the linearized operator of the Swift–Hohenberg equation about $\bar{U}$ and

$$L^\pm := \partial_x^2 \pm \partial_x - \partial_x \bar{U}^{\omega,\varepsilon}.$$

By upper triangular form of (D.6), the eigenvalues of L_ξ, counted by algebraic multiplicity, consist of the union of the eigenvalues of L^0_ξ and $L^\pm_\xi$. Let us first consider the eigenvalues of the Swift–Hohenberg operator L^0_ξ. In [17, 13, 14] (see also [44, 60]) it was analytically verified[34] that for $\varepsilon << 1$, solutions $\bar{U}^{\omega,\varepsilon}$ in (D.5) are spectrally stable for

$$\left|4\omega^2\right| < \frac{1}{3} + \mathcal{O}(\varepsilon)$$

(in particular, for $\omega = 0$). From the fact that the waves are of speed $c \equiv 0$, we find that the characteristic speed of the associated scalar Whitham equation is $a^0 \equiv 0$, and the associated critical mode has expansion $\lambda^0(\xi) = -d^0\xi^2$. Turning to the operators $L^\pm$, and noting that $\bar{U}^{\omega,\varepsilon} \to 0$ uniformly in all derivatives as $\varepsilon \to 0$, we find that as $\varepsilon \to 0$ their eigenvalues approach uniformly the eigenvalues of the limiting constant-coefficient operators

$$\bar{L}^\pm_\xi := (\partial_x + i\xi)^2 \pm (\partial_x + i\xi),$$

which, by direct (discrete Fourier transform) computation, are

$$\bar{\lambda}^\pm(\xi) = -(j+\xi)^2 \pm i(j+\xi),$$

where the Fourier frequency j runs through the integers. By continuity, these are therefore spectrally stable for $|\varepsilon| << 1$, with approximate critical mode expansions (obtained at $j = 0$) of $\pm i\xi - \xi^2$. Combining these facts, we find that the limiting linearized Whitham system has characteristic speeds $a_j = 0, \pm 1$, with corresponding (diagonal) viscosity coefficients d, 1, 1. This verifies (D1)–(D3) and (by distinctness of a_j) (H3) for $|\varepsilon|$ sufficiently small, yielding spectral stability by the discussion of Section D.3 □

[34] It has also been shown numerically that there exist bands of stable periodic Swift–Hohenberg solutions in the parameter space $(\omega, \kappa, \varepsilon)$ [44, 4], for $|\varepsilon|$ not necessarily small.

D.4.5. *Inclined Marangoni flow.* The related inclined thin-film equation

$$H_t + (H^2 - H^3)_x = -(H^3 H_{xxx})_x \tag{D.7}$$

models Marangoni flow driven by a thermal gradient up an inclined silicon wafer, where H denotes fluid height [10, 11]. As a cousin of the Kuramoto–Sivashinsky equation, it would be interesting to investigate whether this model too supports stable periodic traveling-waves solutions.

D.4.6. *Surfactant-driven Marangoni flow.* Finally, we mention the surfactant model [43]

$$\begin{aligned} H_t + \frac{1}{2}(H^2\sigma'(\Gamma)\Gamma_x)_x &= 0, \\ \partial_t\Gamma + \partial_x(\Gamma H\sigma'(\Gamma)\partial_x\Gamma) &= \mathrm{Pe}_{\mathrm{s}}^{-1}\partial_x^4\Gamma, \end{aligned} \tag{D.8}$$

modeling flow in a thin horizontal film driven by surfactant induced gradients in surface tension, where H is fluid height and Γ surface surfactant concentration, and Pe_{s} is the modified Peclet number, a dimensionless constant, and $\sigma(\Gamma) = 1 - \Gamma$ is an equation of state encoding the dependence of surface tension on surfactant density. Like (D.7), this appears to be an interesting example for study by the methods developed here and in [29, 30, 33, 5]. Note, as the second equation is conservative, that the associated Whitham approximation is indeed of system form.

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